Self-consistent treatment of stabilization of resistive wall instabilities in reversed field pinches by radio-frequency waves

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Stabilization of resistive wall magnetohydrodynamic (MHD) instabilities by the force applied by injected radio-frequency (rf) waves is investigated, including the self-consistent effect of the MHD perturbation upon the rf waves in the plasma. This effect leads to the generation of Alfvénic disturbances at the frequency of the rf waves and at the wavelength (in the magnetic surface) of the MHD instability. Stabilization of the ideal external kink instability in the reversed field pinch is considered. If the self-consistent response is neglected, rf waves are confined to the thin vacuum and edge regions, and provide a restoring force which stabilizes the resistive wall instability at moderate rf wave amplitude. However, the generation of Alfvén disturbances causes the rf waves to penetrate deeply into the plasma, eliminating the stabilization. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1416181]

I. INTRODUCTION

The stabilization of magnetohydrodynamic (MHD) instabilities by the ponderomotive force from externally applied rf waves has been studied over the years in various venues. rf stabilization has been investigated for interchange instabilities in mirror machines,[1–4] external kinks in tokamaks[5,6] and Rayleigh–Taylor instabilities in liquid metals (see e.g., Ref. 7). For plasmas, it generally appears that stabilization requires applied rf fields of large amplitude.

In the past analysis of rf stabilization of external modes the self-consistent effect of the MHD perturbation on the rf waves has not been fully included. We include this effect here, and find that self-consistency has a large effect on the rf stabilization. We solve the problem for a particular case of interest—resistive wall instabilities in the reversed field pinch.

The resistive wall modes are potentially dangerous instabilities in long-pulse reversed field pinch (RFP) experiments. Several techniques have been proposed to stabilize these modes in RFPs. One of them is the development of an active feedback stabilization systems (see e.g., Ref. 8). The other is the introduction of a rotating wall (see e.g., Ref. 9). If successful, the rf stabilization would be an attractive alternative to these stabilizing methods.

In the reversed field pinch an ideal external kink mode is unstable in the absence of a conducting shell. In the presence of a shell of finite resistivity the mode is still unstable, although its growth time is slowed approximately to the electrical penetration time of the shell. The resistive wall instability—a mode which becomes unstable when a perfectly conducting shell is replaced by a resistive shell—is also important in other configurations such as advanced tokamaks and spherical tokamaks.

We consider that the plasma is surrounded by a thin vacuum region, which is bounded by a resistive wall. Although the magnetic field of the growing MHD mode penetrates the wall, the higher frequency rf waves do not significantly penetrate the wall. For the rf waves the wall appears essentially as a perfect conductor. Hence, we may expect that rf wave energy density trapped between the highly conducting plasma and the wall will provide a restoring force on the MHD perturbation of the plasma surface. This picture will roughly apply if the rf waves do not penetrate deeply into the plasma.

For a RFP the magnetic field at the plasma edge is mainly poloidal. We consider the application of a parallel (poloidal) rf electric field with poloidal mode number $m=0$. Such an electric field polarization will not generate plasma flow or plasma waves in the equilibrium plasma. It will simply decay resistively into the plasma. The $m=0$ wave can be applied through an oscillating poloidal surface loop voltage. This may be practical if the rf wave frequency is much less than the ion cyclotron frequency. This simple picture holds true if we consider that the dielectric response of the rf waves to the MHD perturbation is that of an unmagnetized plasma. In this case, the rf waves evanesce into the plasma a resistive skin depth. For realistic conditions the skin depth is sufficiently small that the rf energy, concentrated to a small region, is strongly altered by the MHD perturbation and stabilizes the external kink resistive wall instability for rf wave of moderate amplitude (wave magnetic field much less than the equilibrium field). This is quite similar to the stabilization of the Rayleigh–Taylor instability in liquid metals.

If we consider the plasma dielectric properties to be that of a magnetized plasma, the dynamics becomes more interesting. The effect of the MHD perturbation of the surface on the rf waves leads to the generation of Alfvénic disturbances. The Alfvén disturbances are characterized by the frequency of the rf waves and the wave number (within the magnetic surface) of the MHD perturbation. The perturbed surface generates two types of disturbances. A decaying wave which is a variant of a shear Alfvén wave is generated. Were this the only response then the rf waves would be confined to the

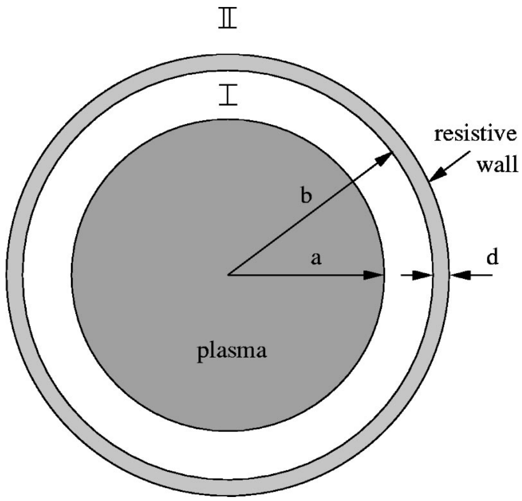

FIG. 1. Geometry for the resistive wall stability analysis.

edge and vacuum region, yielding a strong stabilizing influence. However, compressional waves which propagate into the plasma are also generated. The rf energy is no longer trapped to the edge, and the stabilization vanishes.

The case of the unmagnetized dielectric response is presented in Sec. II (where the perturbed rf pressure is calculated) and Sec. III (where the growth rates are calculated). The more realistic case of the magnetized plasma is discussed in Sec. IV. We summarize in Sec. V.

II. SIMPLIFIED CONSIDERATION OF RF PRESSURE

We consider a cylindrical geometry shown in Fig. 1 with the plasma radius a and the vessel radius b. A rf voltage of frequency ω is applied to the toroidal gap (horizontal insulated cut in the conducting shell). We can assume that in this model the rf waves are excited by a uniformly distributed electric field on the surface of the vessel,

$$\mathbf{E}(b)=\tfrac{1}{2}(E_A e^{-i\omega t}+\text{c.c.})\mathbf{e}_\theta.$$

In this simplified consideration we assume the dielectric properties of plasma to be that of an unmagnetized, perfectly conducting plasma. In the frequency ranges of interest $\omega \ll \omega_{ci}$ or $\omega \gtrsim \omega_{ci}$,

$$\lambda=\frac{\omega a}{c}\ll 1. \tag{1}$$

First we find the unperturbed em fields. From Maxwell's equations we find that the nonzero field components satisfying the boundary condition $E_\theta(a)=0$ are

$$E_\theta=\frac{\delta E_A}{\delta^2-1}\,\frac{(r-a)(r+a)}{ar}, \tag{2}$$

$$B_z=\frac{2}{i\lambda}\,\frac{\delta E_A}{\delta^2-1}, \tag{3}$$

where $\delta=b/a$ and Eq. (1) is used. The time averaged rf pressure on the plasma surface is

$$P_0=\frac{|\mathbf{B}|^2}{16\pi}=\frac{1}{16\pi}\,\frac{4}{\lambda^2}\left(\frac{\delta}{\delta^2-1}\right)^2|E_A|^2.$$

In the frequency range under consideration we neglect the contribution to the rf pressure from the component of electric field normal to the plasma surface.

Now using a perturbative approach we find the rf pressure on a perturbed plasma surface. Consider a surface perturbation of the form,

$$r_s=a+\tfrac{1}{2}(Ae^{im\theta+ikz}+\text{c.c.}). \tag{4}$$

Then including the terms of the first order in A, the two unit vectors tangential to the perturbed surface are

$$\boldsymbol{\tau}_1=\tfrac{1}{2}\left(i\frac{m}{a}Ae^{im\theta+ikz}+\text{c.c.}\right)\mathbf{e}_r+\mathbf{e}_\theta,$$

$$\boldsymbol{\tau}_2=\tfrac{1}{2}(ikAe^{im\theta+ikz}+\text{c.c.})\mathbf{e}_r+\mathbf{e}_z.$$

For a perturbed plasma surface, the rf fields in the vacuum layer are a superposition of the unperturbed fields of Eqs. (2), (3) and a perturbation. The amplitudes of perturbed rf fields satisfying Maxwell's equations are

$$E_z=[B^+I_m(|k|r)+C^+K_m(|k|r)]e^{im\theta+ikz}+[B^-I_m(|k|r)+C^-K_m(|k|r)]e^{-im\theta-ikz},$$

$$B_z=[D^+I_m(|k|r)+F^+K_m(|k|r)]e^{im\theta+ikz}+[D^-I_m(|k|r)+F^-K_m(|k|r)]e^{-im\theta-ikz},$$

$$B_\theta=-\frac{i\omega}{c|k|}[B^+I_m'(|k|r)+C^+K_m'(|k|r)]e^{im\theta+ikz}+\frac{m}{rk}[D^+I_m(|k|r)+F^+K_m(|k|r)]e^{im\theta+ikz}+\text{``}-\text{,''}$$

$$E_\theta=\frac{m}{rk}[B^+I_m(|k|r)+C^+K_m(|k|r)]e^{im\theta+ikz}+\frac{i\omega}{c|k|}[D^+I_m'(|k|r)+F^+K_m'(|k|r)]e^{im\theta+ikz}+\text{``}-\text{,''}$$

where the terms with "$-$" are obtained from the terms with "$+$" by changing $m\to -m$ and $k\to -k$, and the order m of the Bessel functions is positive (subscript m refers to $|m|$). Also it is assumed that $\omega/c\ll|k|$ which is consistent with the condition of Eq. (1).

The boundary conditions $E_\theta(b)=0$ and $E_z(b)=0$ give

$$C^+=-\frac{I_m(|k|b)}{K_m(|k|b)}B^+,\quad F^+=-\frac{I_m'(|k|b)}{K_m'(|k|b)}D^+. \tag{5}$$

The same relations hold for the coefficients with "$-$."

The boundary conditions on the plasma surface are $\mathbf{E}(r_s)\cdot\boldsymbol{\tau}_{1,2}=0$. Including terms of first order in A, one obtains

$$2\tilde{E}_A\frac{1}{2}A+\frac{m}{k}f_{\text{rf}}B^++\frac{i\omega}{c|k|}g_{\text{rf}}'D^+=0, \tag{6}$$

$$B^+I_m(|k|a)+C^+K_m(|k|a)=0, \tag{7}$$

where $\tilde{E}_A=\delta E_A/(\delta^2-1)$ and

$$f_{\rm rf}=I_m(|k|a)-\frac{I_m(|k|b)}{K_m(|k|b)}K_m(|k|a),$$

$$g_{\rm rf}=I_m(|k|a)-\frac{I'_m(|k|b)}{K'_m(|k|b)}K_m(|k|a),$$

$$g'_{\rm rf}=I'_m(|k|a)-\frac{I'_m(|k|b)}{K'_m(|k|b)}K'_m(|k|a).$$

The equation for B^- and D^- is obtained by changing $A\rightarrow A^*$ in Eq. (6).

We solve the Eqs. (5)–(7) and find the perturbed amplitude

$$B_z=\frac{g_{\rm rf}}{g'_{\rm rf}}\frac{2i\tilde{E}_A|k|}{\lambda}\frac{1}{2}(Ae^{im\theta+ikz}+\text{c.c.}).$$

Then the total time averaged rf pressure on the plasma surface with the linear accuracy in A is

$$P=\frac{1}{16\pi}\left|\frac{2}{i\lambda}\tilde{E}_A+\frac{g_{\rm rf}}{g'_{\rm rf}}\frac{2i\tilde{E}_A|k|}{\lambda}\frac{1}{2}(Ae^{im\theta+ikz}+\text{c.c.})\right|^2$$

$$\approx P_0+\tilde{p}_{\rm rf}\frac{1}{2}(Ae^{im\theta+ikz}+\text{c.c.}), \tag{8}$$

with

$$\tilde{p}_{\rm rf}=-P_0\frac{g_{\rm rf}}{g'_{\rm rf}}2|k|>0.$$

Since $\tilde{p}_{\rm rf}>0$, the perturbed rf pressure profile has a stabilizing effect on the plasma surface.

In the present analysis the unperturbed rf pressure P_0 is much smaller than the typical magnetic pressure in the plasma. It is assumed that the influence of rf pressure on the plasma equillibrium is negligible.

Although small, the rf pressure can influence the stability properties of the plasma. From Eq. (8) it follows that the perturbed rf pressure is proportional to $A/(b-a)$ ($g'_{\rm rf}\propto b-a$), while the MHD force acting on the perturbed plasma column is proportional to A/a (for the low frequencies of resistive wall modes). Thus for a sufficiently thin vacuum layer the perturbed MHD and rf forces can be comparable even when the unperturbed rf pressure is significantly smaller than the magnetic pressure, which may lead to an effective stabilization of external modes in RFPs.

III. CALCULATION OF THE GROWTH RATES

We follow closely the derivation of the growth rates of the resistive wall modes given in Chapter 9 in Ref. 10 for the general screw pinch. We modify the pressure balance equation on the plasma–vacuum boundary by including the rf pressure calculated in the previous section.

The length of the cylinder is $2\pi R_0$. The equilibrium magnetic field is given by $\mathbf{B}=B_\theta(r)\mathbf{e}_\theta+B_z(r)\mathbf{e}_z$ and the perturbations are of the form $\boldsymbol{\xi}(\mathbf{r})=\boldsymbol{\xi}(r)\exp[i(m\theta+kz)]$. The displacement is decomposed as $\boldsymbol{\xi}=\xi\mathbf{e}_r+\eta\mathbf{e}_\eta+\xi_\parallel\mathbf{b}$, where $\mathbf{e}_\eta=(B_z\mathbf{e}_\theta-B_\theta\mathbf{e}_z)/B$ and $\mathbf{b}$ is the unit vector in the direction of $\mathbf{B}$.

In the vacuum region I (see Fig. 1) the perturbed MHD magnetic field amplitudes (separate from the perturbed rf fields) are

$$B_{1r}=k[c_1I'_m(|k|r)+c_2K'_m(|k|r)],$$

$$B_{1\theta}=\frac{im}{r}[c_1I_m(|k|r)+c_2K_m(|k|r)],$$

$$B_{1z}=ik[c_1I_m(|k|r)+c_2K_m(|k|r)].$$

In the vacuum region II the amplitudes are

$$B_{1r}=kc_5K'_m(|k|r),\quad B_{1\theta}=\frac{im}{r}c_5K_m(|k|r),$$

$$B_{1z}=ikc_5K_m(|k|r).$$

Within the resistive wall $\mathbf{B}_1$ satisfies a magnetic diffusion equation,

$$\frac{\partial\mathbf{B}_1}{\partial t}=\frac{\eta c^2}{4\pi}\nabla^2\mathbf{B}_1.$$

This equation is solved in a thin wall limit $d\ll b$. Since the growth rate γ will be of order $\gamma\sim1/\tau_D=\eta c^2/4\pi bd$, then $4\pi\gamma/\eta c^2\sim1/bd\gg k^2+m^2/b^2$ and

$$B_{1r}=c_3e^{\lambda_1(r-b)}+c_4e^{-\lambda_1(r-b)},$$

with $\lambda_1^2=4\pi\gamma/\eta c^2$. The remaining components of $\mathbf{B}_1$ are given by the two relations,

$$\frac{m}{b}B_{1z}-kB_{1\theta}=0,$$

$$i\left(\frac{m}{b}B_{1\theta}+kB_{1z}\right)=-\lambda_1[c_3e^{\lambda_1(r-b)}-c_4e^{-\lambda_1(r-b)}].$$

The first equation follows from tangential continuity arguments and the fact that $(m/r)B_{1z}=kB_{1\theta}$ in both regions I and II. The second equation is a consequence of $\nabla\cdot\mathbf{B}_1=0$.

To find a dispersion relation for γ, we match the solutions in each region by applying the boundary conditions. At the boundary of the regions I, II and the wall one of the tangential components of $\mathbf{B}_1$ and the normal components of $\mathbf{B}_1$ are continuous (due to the above equation the second tangential component is continuous automatically).

Across the region I-plasma interface the boundary conditions require continuity in the normal component of magnetic field $[B_{1r}]=0$ and the perpendicular pressure balance

$$\left[p_1+\frac{\mathbf{B}\mathbf{B}_1}{4\pi}+\boldsymbol{\xi}\nabla\left(p+\frac{B^2}{8\pi}\right)\right]=0.$$

To evaluate these conditions it is necessary to express each of the plasma quantities in terms of the value of ξ on the boundary.

The perturbed magnetic field $\mathbf{B}_1$ can be found from MHD equations in terms of $\boldsymbol{\xi}$,

$$B_{1r}=i\left(\frac{m}{r}B_\theta+kB_z\right)\xi,$$

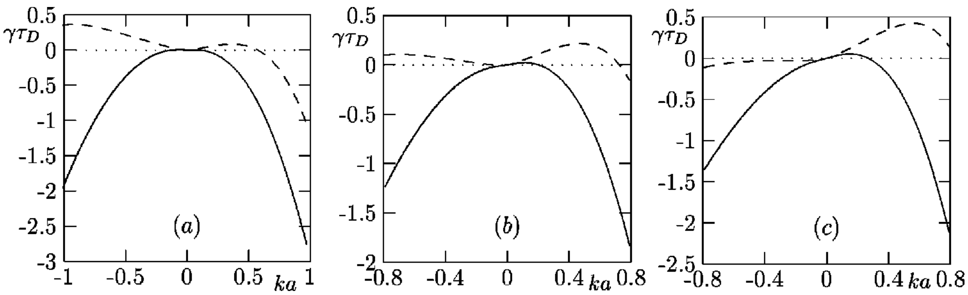

FIG. 2. $\gamma\tau_D$ vs ka. (a) $\mu a=2.4$, (b) $\mu a=2.5$, (c) $\mu a=2.6$. In all cases: (---) $P_0/P_B=0$, (—) $P_0/P_B=0.05$; $b/a=1.1$, $m=1$.

$$B_{1\theta}=ikB\eta-(\xi B_\theta)',$$

$$B_{1z}=-\frac{1}{r}(r\xi B_z)'-\frac{im}{r}B\eta.$$

In the low frequency limit the component η is related to ξ (see Appendix C in Ref. 10) by

$$\eta=\frac{i}{rBk_0^2}[G(r\xi)'+2kB_\theta\xi],$$

where $k_0^2=k^2+m^2/r^2$, $G=mB_z/r-kB_\theta$. For simplicity we assume that no surface currents are present on the plasma–vacuum interface and that the plasma pressure decays smoothly to zero at the plasma edge. Then the pressure balance becomes

$$\mathbf{B}_a\mathbf{B}_{1a}^p=\mathbf{B}_a\mathbf{B}_{1a}^v+4\pi\tilde{p}_{\mathrm{rf}}\xi_a,$$

where $\tilde{p}_{\mathrm{rf}}$ is defined in Eq. (8).

Then the boundary conditions across the region I-plasma interface can be written as

$$k[c_1I_m'(|k|a)+c_2K_m'(|k|a)]=iF_a\xi_a,$$

$$iF_a[c_1I_m(|k|a)+c_2K_m(|k|a)]+4\pi\tilde{p}_{\mathrm{rf}}\xi_a$$

$$=-\frac{F_a}{ak_0^2}(\hat{F}_a\xi_a+F_aa\xi_a'),$$

where $F=(m/r)B_\theta+kB_z$, $\hat{F}=kB_z-(m/r)B_\theta$.

Combining these equations with the other four boundary conditions, after some calculations one finds

$$\gamma\tau_D=\frac{k^2b^2+m^2}{k^2b^2K_m'(|k|b)I_m'(|k|b)\left[1-\frac{I_m'(|k|a)K_m'(|k|b)}{I_m'(|k|b)K_m'(|k|a)}\right]}\times\frac{\delta W_\infty+\delta W_{\mathrm{rf}}}{\delta W_b+\delta W_{\mathrm{rf}}}, \tag{9}$$

where $\tau_D=4\pi bd/\eta c^2$,

$$\delta W_{\mathrm{rf}}=2\pi^2R_0a\tilde{p}_{\mathrm{rf}}\xi_a^2, \tag{10}$$

$$\frac{\delta W_\infty}{(\pi/2)R_0}=\int_0^a(f\xi'^2+g\xi^2)dr+\left[\left(\frac{krB_z-mB_\theta}{k_0^2r^2}\right)rF+\frac{r^2\Lambda_\infty F^2}{|m|}\right]_a\xi_a^2, \tag{11}$$

$$\frac{\delta W_b}{(\pi/2)R_0}=\frac{\delta W_\infty}{(\pi/2)R_0}+\frac{a^2F_a^2}{|m|}(\Lambda_b-\Lambda_\infty)\xi_a^2, \tag{12}$$

where

$$\Lambda_b-\Lambda_\infty=\Lambda_\infty\frac{I_m'(|k|a)/K_m'(|k|a)-I_m(|k|a)/K_m(|k|a)}{I_m'(|k|b)/K_m'(|k|b)-I_m'(|k|a)/K_m'(|k|a)},$$

$$\Lambda_\infty=-\frac{|m|K_m(|k|a)}{|k|aK_m'(|k|a)},$$

$$f=\frac{rF^2}{k_0^2},$$

$$g=\left(\frac{k_0^2r^2-1}{k_0^2r^2}\right)rF^2+2\frac{k^2}{rk_0^4}\left(kB_z-\frac{mB_\theta}{r}\right)F.$$

δW_{rf} defined in Eq. (10) is proportional to the work done by the forces of rf pressure exerted on the plasma–vacuum surface when this surface is continuously perturbed by increasing ξ_a from 0 to its amplitude value. δW_∞ and δW_b relate to δW with a perfectly conducting wall, located at $r=\infty$ and $r=b$ correspondingly. In the above equation for g we assumed zero plasma pressure.

The function $\xi(r)$ satisfies a differential equation of second order with the regularity conditions at $r=0$. This means that $\xi(r)$ is defined by the boundary condition $\xi(a)=\xi_a$. For a mode unstable without the rf pressure, $\delta W_\infty<0$ and $\delta W_b>0$. In Eq. (9) one can assume that $\delta W_{\mathrm{rf}}\ll\delta W_b$. Because $\delta W_{\mathrm{rf}}>0$, the rf pressure is a stabilizing influence.

To estimate the changes to the growth rates, we consider the Bessel function equilibrium (the Taylor state) given by

$$\frac{B_z}{B_0}=J_0(\mu r),\quad \frac{B_\theta}{B_0}=J_1(\mu r),$$

where B_0 is the toroidal field on axis, J_0 and J_1 are Bessel functions. For this state B_z reverses direction with radius if $\mu a>2.4$.

For the estimation of δW_∞ given by Eq. (11) instead of an exact function $\xi(r)$ we take the trial function $\xi(r)=\xi_a$, $0\leq r\leq a$. This choice is appropriate for an analysis of external modes. With these assumptions the growth rate $\gamma\tau_D$ is defined by the dimensionless parameters P_0/P_B ($P_B=B_0^2/8\pi$), μa, b/a, m, ka. The modes unstable without rf power are only with $m=\pm1$. Here we consider only these modes. Because the growth rate is symmetric with respect to simultaneous change $m\to-m$, $k\to-k$, we consider the modes with $m=1$.

We calculate $\gamma\tau_D$ for several values of the parameter μa corresponding to equilibria with field reversal. Figures 2(a)–(c) show the dependence of $\gamma\tau_D$ vs ka for $\mu a=2.4$; 2.5; 2.6. On these figures $b/a=1.1$, $m=1$. The

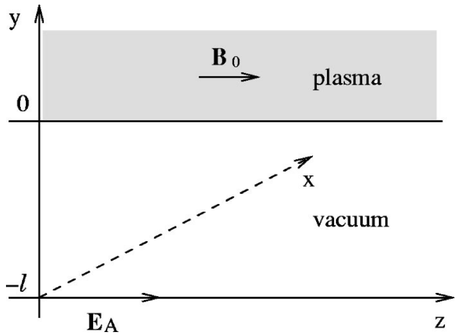

FIG. 3. Geometry for a magnetized plasma analysis.

dashed lines are $\gamma\tau_D$ without rf pressure and the solid lines are $\gamma\tau_D$ with $P_0/P_B=0.05$. When $\gamma>0$ the mode is unstable, and when $\gamma<0$ the mode is stable.

From this figure one can see that without the rf pressure the mode is unstable for some range of wave numbers ka. With the rf pressure the mode is either stabilized or its maximum growth rate is reduced by approximately an order of magnitude. If one identifies an equivalent torus of length $2\pi R_0$, then the wave number k becomes quantized: $ka=na/R_0$. If we take the aspect ratio $R_0/a=3$, then from Fig. 2 one can see that the mode is stabilized by the rf pressure for all wave numbers.

This analysis shows that the stabilization of the resistive wall modes in RFPs is possible with application of moderate rf pressure. From this point of view this approach seems to be attractive. Further analysis with a more realistic description of the dielectric properties of the plasma is necessary, however, in order to make a more substantiated conclusion about the applicability of this approach.

IV. RF PRESSURE ON A MAGNETIZED PLASMA

A. Magnetized plasma with a skin layer, $\omega_{\rm rf}\ll\omega_{ci}$

In this section we find the distribution of rf pressure on a perturbed boundary of a magnetized plasma. We consider a plane geometry with a uniform magnetic field $\mathbf{B}_0$ directed along the z axis (see Fig. 3). The width of vacuum layer is l. rf electric field is applied at the wall in the direction parallel to the magnetic field in the plasma,

$$E_z=\tfrac{1}{2}(E_A e^{-i\omega t}+\text{c.c.}).$$

Here we examine the rf frequency range $\omega\ll\omega_{ci}$ such that the propagation of em waves in plasma is described by a resistive MHD model. We also ignore plasma pressure in this model and assume that the plasma density is uniform.

The rf fields of interest penetrate into plasma on a depth of the skin layer. Therefore it is reasonable to assume the uniformity of the magnetic field and of the plasma density. The rf pressure on the plasma surface is obtained by integrating the volume density of rf force across the skin layer.

The sharp plasma boundary model is used here for simplicity. However we do not expect a significant change of the result if the plasma density is allowed to decay smoothly to zero as long as the length of penetration of rf fields into the plasma is much smaller than the width of the vacuum layer. Also it is assumed that in the vacuum region the plasma density and temperature are small enough so that the plasma dielectric properties in this region are that of a vacuum.

We first find em fields in the unperturbed plasma. Linearized equations for rf fields in the unperturbed plasma are

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}=\frac{1}{c}\mathbf{j}\times\mathbf{B}_0,\quad \mathbf{E}+\frac{\boldsymbol{v}}{c}\times\mathbf{B}_0=\eta\mathbf{j},$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}.$$

We find the solution in the form,

$$\mathbf{E}=\tfrac{1}{2}(\mathbf{E}e^{-i\omega t}+\text{c.c.}).$$

Then the solution in the plasma is

$$\boldsymbol{v}=0,$$

$$E_{zp}=\alpha\frac{E_A}{\sin\lambda+\alpha\cos\lambda}e^{ik_{y0}y},$$

$$B_{xp}=\frac{iE_A}{\sin\lambda+\alpha\cos\lambda}e^{ik_{y0}y},$$

$$j_z=\frac{c}{4\pi}\frac{k_{y0}E_A}{\sin\lambda+\alpha\cos\lambda}e^{ik_{y0}y},$$

and in vacuum is

$$E_z=[-\sin(\omega y/c)+\alpha\cos(\omega y/c)]\frac{E_A}{\sin\lambda+\alpha\cos\lambda},$$

$$B_x=[\cos(\omega y/c)+\alpha\sin(\omega y/c)]\frac{iE_A}{\sin\lambda+\alpha\cos\lambda},$$

where

$$\lambda=\omega l/c,\quad \alpha=\sqrt{\frac{\omega\eta}{8\pi}}(1+i),\quad k_{y0}=\sqrt{\frac{4\pi\omega}{2\eta c^2}}(1+i).$$

Then the unperturbed rf pressure on the plasma boundary is given by

$$P_0=\int_0^\infty\frac{1}{c}\langle j_zB_x\rangle dy=\frac{\beta\beta^*}{16\pi},$$

where $\beta=E_A/(\sin\lambda+\alpha\cos\lambda)$.

If the resistivity η is found from the electron–ion collision rates, then for the boundary plasma with $n\sim5\cdot10^{12}\,\text{cm}^{-3}$, $T\sim40\,\text{eV}$, $\omega\sim10^5$ rad/s the skin depth corresponding to the wave number k_{y0} is ~1 cm.

We assume that the perturbation of the plasma–vacuum boundary,

$$y_s=\tfrac{1}{2}(Ae^{ik_xx+ik_zz}+\text{c.c.})$$

is much smaller than the skin depth, and also that

$$|k_x|,|k_z|\ll|k_{y0}|. \tag{13}$$

The normal vector to the perturbed surface is

$$\mathbf{n}=\tfrac{1}{2}(-ik_xAe^{ik_xx+ik_zz}+\text{c.c.})\mathbf{e}_x+\mathbf{e}_y+\tfrac{1}{2}(-ik_zAe^{ik_xx+ik_zz}+\text{c.c.})\mathbf{e}_z.$$

The total MHD magnetic field is

$$\mathbf{B}=B_0\mathbf{e}_z+\tfrac{1}{2}(\mathbf{B}e^{ik_xx+ik_yy+ik_zz}+\text{c.c.}). \quad (14)$$

Since the perturbed plasma–vacuum boundary stays on the perturbed magnetic surface, then $\mathbf{B\cdot n}=0$ at $y=0$, which gives

$$B_y=ik_zAB_0. \quad (15)$$

We consider single-fluid resistive MHD equations without pressure and express each quantity in the form,

$$f=f_0+f_{0\text{rf}}+f+f_{\text{rf}},$$

where f_0 is the unperturbed time-independent (equilibrium) quantity and $f_{0\text{rf}}$ is the unperturbed quantity varying with frequency ω. The terms f and f_{rf} are the perturbations to the equilibrium and rf fields, respectively.

The unperturbed fields are

$$\boldsymbol{v}_0=0,\ \ \boldsymbol{v}_{0\text{rf}}=0,\ \ \rho_0,\ \ \rho_{0\text{rf}}=0,\ \ \mathbf{j}_0=0,\ \ \mathbf{j}_{0\text{rf}},$$

$$\mathbf{B}_0=B_0\mathbf{e}_z,\ \ \mathbf{B}_{0\text{rf}},\ \ \mathbf{E}_0=0,\ \ \mathbf{E}_{0\text{rf}}.$$

We perform separation of fast (rf) and slow (growth) time scales. The linearized momentum equation for rf quantities is

$$\rho_0\frac{\partial\boldsymbol{v}_{\text{rf}}}{\partial t}=\frac{1}{c}(\mathbf{j}_{0\text{rf}}\times\mathbf{B}+\mathbf{j}\times\mathbf{B}_{0\text{rf}}+\mathbf{j}_{\text{rf}}\times B_0\mathbf{e}_z)$$

and the equation for the volume density of rf force is

$$\mathbf{f}=\frac{1}{c}\langle\mathbf{j}_{0\text{rf}}\times\mathbf{B}_{0\text{rf}}+\mathbf{j}_{0\text{rf}}\times\mathbf{B}_{\text{rf}}+\mathbf{j}_{\text{rf}}\times\mathbf{B}_{0\text{rf}}\rangle. \quad (16)$$

From the linearized Ohm's Law the equation for the rf fields is

$$\mathbf{E}_{\text{rf}}+\frac{1}{c}\boldsymbol{v}_{\text{rf}}\times B_0\mathbf{e}_z=\eta\mathbf{j}_{\text{rf}}.$$

Where we have taken into account that $|\boldsymbol{v}\times\mathbf{B}_{0\text{rf}}|\ll|\boldsymbol{v}_{\text{rf}}\times B_0\mathbf{e}_z|$.

Since the resultant equations are linear with respect to rf fields, we can consider equations for their complex amplitudes assuming the time dependence $\propto\exp(-i\omega t)$. Then after substituting $\boldsymbol{v}_{\text{rf}}$ found from the momentum equation into Ohm's Law and using $\eta\omega\ll v_A^2/c^2$, we find

$$j_{x\text{rf}}=\frac{c^2\omega\rho_0}{iB_0^2}E_{x\text{rf}}+\frac{1}{B_0}(B_xj_{z0\text{rf}}+B_{x0\text{rf}}j_z),$$

$$j_{y\text{rf}}=\frac{c^2\omega\rho_0}{iB_0^2}E_{y\text{rf}}+\frac{B_y}{B_0}j_{z0\text{rf}},$$

$$j_{z\text{rf}}=\frac{1}{\eta}E_{z\text{rf}}. \quad (17)$$

In the first equation the term $(B_{x0\text{rf}}/B_0)j_z$ can be neglected when compared with the term $(B_y/B_0)j_{z0\text{rf}}$ in the second equation. The neglect of this term is accurately justified when the fields calculated without this term are substituted into the original Eqs. (17).

Substituting these current components and Eq. (14) into the Maxwell's equations, we find

$$\nabla^2E_x+\frac{\omega^2}{v_A^2}E_x-\frac{\partial}{\partial x}\nabla\cdot\mathbf{E}=-\frac{i\omega k_{y0}\beta}{c}\frac{1}{2}\left(\frac{B_x}{B_0}e^{ik_xx+ik_yy+ik_zz}+\text{c.c.}\right)e^{ik_{y0}y},$$

$$\nabla^2E_y+\frac{\omega^2}{v_A^2}E_y-\frac{\partial}{\partial y}\nabla\cdot\mathbf{E}=-\frac{i\omega k_{y0}\beta}{c}\frac{1}{2}\left(\frac{B_y}{B_0}e^{ik_xx+ik_yy+ik_zz}+\text{c.c.}\right)e^{ik_{y0}y},$$

$$\nabla^2E_z+\frac{4\pi i\omega}{\eta c^2}E_z-\frac{\partial}{\partial z}\nabla\cdot\mathbf{E}=0. \quad (18)$$

In Eqs. (18) we omitted the subscript "rf" in the electric field components on the left-hand side.

We look for a particular solution of the system of Eqs. (18) in the form,

$$\tilde{\mathbf{E}}=\tilde{\mathbf{E}}^+e^{ik_xx+ik_{y0}y+ik_zz}+\tilde{\mathbf{E}}^-e^{-ik_xx+ik_{y0}y-ik_zz},$$

where we assumed for simplicity that $k_y=0$. The particular solution incorporates the effect of the MHD perturbation on the rf waves. Then for the amplitudes with "+," Eqs. (18) become

$$-k^2E_x+\frac{\omega^2}{v_A^2}E_x+k_x(k_xE_x+k_{y0}E_y+k_zE_z)=-\frac{i\omega k_{y0}\beta}{c}\frac{1}{2}\frac{B_x}{B_0},$$

$$-k^2E_y+\frac{\omega^2}{v_A^2}E_y+k_{y0}(k_xE_x+k_{y0}E_y+k_zE_z)=-\frac{i\omega k_{y0}\beta}{c}\frac{1}{2}\frac{B_y}{B_0}, \quad (19)$$

$$-k^2E_z+\frac{4\pi i\omega}{\eta c^2}E_z+k_z(k_xE_x+k_{y0}E_y+k_zE_z)=0,$$

with $k^2=k_x^2+k_{y0}^2+k_z^2$.

Solving Eqs. (19) and the corresponding equations for the amplitudes with "−," we find the particular solution of the nonuniform system of Eqs. (18),

$$\tilde{E}_x=\frac{i\omega\beta k_x}{2cB_0k_{y0}}\frac{B_xk_x\left(-k_x^2+\frac{\omega^2}{v_A^2}+k_{y0}^2\frac{k_z^2}{k_x^2}\right)-B_yk_{y0}(k_x^2+k_z^2)}{k_{y0}^2k_z^2+\frac{\omega^2}{v_A^2}k_x^2}\times e^{ik_xx+ik_{y0}y+ik_zz}+\text{"}-\text{,"} \quad (20)$$

$$\tilde{E}_z=-\frac{i\omega\beta k_{y0}k_z}{2cB_0}\frac{B_xk_x+B_yk_{y0}}{k_{y0}^2k_z^2+\frac{\omega^2}{v_A^2}k_x^2}e^{ik_xx+ik_{y0}y+ik_zz}+\text{"}-\text{,"} \quad (21)$$

$$\tilde{B}_x = -\frac{i\beta k_{y0}^2 k_z}{2B_0}\frac{B_x k_x + B_y k_{y0}}{k_{y0}^2 k_z^2 + \frac{\omega^2}{v_A^2}k_x^2} e^{ik_x x + ik_{y0}y + ik_z z} + \text{``}-,\text{''} \tag{22}$$

$$\tilde{B}_z = -\frac{i\beta}{2B_0}\frac{B_x\left(k_{y0}^2 k_z^2 + \frac{\omega^2}{v_A^2}k_x^2\right) - B_y k_x k_z^2 k_{y0}}{k_{y0}^2 k_z^2 + \frac{\omega^2}{v_A^2}k_x^2} \times e^{ik_x x + ik_{y0}y + ik_z z} + \text{``}-.\text{''} \tag{23}$$

In the above equations the terms with "$-$" are obtained from those with "+" by changing $k_x \to -k_x$, $k_z \to -k_z$ and $B_x \to B_x^*$, $B_y \to B_y^*$. In the derivation of the above equations and in the following calculations the condition of Eq. (13) is used to simplify the analysis.

Now we find the general solution of Eqs. (18) without the right-hand side. We find this solution in the form,

$$\mathbf{E} = \mathbf{E}e^{ik_x x + ik_y y + ik_z z},$$

the wave number k_y is now different from that in MHD magnetic field perturbation (which we have zeroed). Then Eqs. (18) without the right-hand side become

$$\left(k_x^2 - k^2 + \frac{\omega^2}{v_A^2}\right)E_x + k_x k_y E_y + k_x k_z E_z = 0,$$

$$k_x k_y E_x + \left(k_y^2 - k^2 + \frac{\omega^2}{v_A^2}\right)E_y + k_y k_z E_z = 0, \tag{24}$$

$$k_x k_z E_x + k_y k_z E_y + \left(k_z^2 - k^2 + \frac{i\omega^2}{\tilde{\eta}c^2}\right)E_z = 0,$$

where $\tilde{\eta} = \omega\eta/4\pi$. The Eqs. (24) lead to a dispersion equation which has two roots for k_y^2 when k_x, k_z, and ω are fixed. These roots are

$$k_{y1}^2 = -k_x^2 - k_z^2 + \frac{\omega^2}{v_A^2},$$

$$k_{y2}^2 = -k_x^2 + k_{y0}^2 - \frac{iv_A^2}{\tilde{\eta}c^2}k_z^2.$$

The relations between the field components corresponding to each root are found from Eqs. (24). The root k_{y1} corresponds to the compressional Alfvén wave propagating from the boundary toward the plasma. This wave is excited by the perturbation. For this branch

$$E_y = -\frac{k_x}{k_{y1}}E_x, \quad E_z = 0,$$

$$B_x = \frac{c}{\omega}\frac{k_x k_z}{k_{y1}}E_x, \quad B_y = \frac{c}{\omega}k_z E_x, \quad B_z = -\frac{c}{\omega}\frac{k_x^2 + k_{y1}^2}{k_{y1}}E_x.$$

The root k_{y2} corresponds to a decaying wave, localized approximately in the skin layer, for which

$$E_x = \frac{k_x k_z}{k_z^2 - \frac{\omega^2}{v_A^2}}E_z, \quad E_y = \frac{k_{y2}k_z}{k_z^2 - \frac{\omega^2}{v_A^2}}E_z,$$

$$B_x = -\frac{c}{\omega}\frac{k_{y2}\frac{\omega^2}{v_A^2}}{k_z^2 - \frac{\omega^2}{v_A^2}}E_z, \quad B_y = \frac{c}{\omega}\frac{k_x\frac{\omega^2}{v_A^2}}{k_z^2 - \frac{\omega^2}{v_A^2}}E_z, \quad B_z = 0.$$

This branch is a modification of the shear Alfvén wave due to finite η and E_z.

Now the general solution of the uniform part of Eqs. (18) is

$$E_x = E_{x1}^+ e^{ik_{y1}y + ik_x x + ik_z z} + \frac{k_x k_z}{k_z^2 - \frac{\omega^2}{v_A^2}}E_{z2}^+ e^{ik_{y2}y + ik_x x + ik_z z} + \text{``}-,\text{''}$$

$$E_z = E_{z2}^+ e^{ik_{y2}y + ik_x x + ik_z z} + \text{``}-,\text{''}$$

$$B_x = \frac{c}{\omega}\frac{k_x k_z}{k_{y1}}E_{x1}^+ e^{ik_{y1}y + ik_x x + ik_z z} - \frac{c}{\omega}\frac{k_{y2}\frac{\omega^2}{v_A^2}}{k_z^2 - \frac{\omega^2}{v_A^2}}E_{z2}^+ e^{ik_{y2}y + ik_x x + ik_z z} + \text{``}-,\text{''}$$

$$B_z = -\frac{c}{\omega}\frac{k_x^2 + k_{y1}^2}{k_{y1}}E_{x1}^+ e^{ik_{y1}y + ik_x x + ik_z z} + \text{``}-.\text{''}$$

The terms with "$-$" are obtained from those with "+" by changing $k_x \to -k_x$, $k_z \to -k_z$. In these equations the coefficients $E_{x1}^{\pm}$ and $E_{z2}^{\pm}$ are arbitrary.

Perturbed fields in the vacuum layer, satisfying the boundary conditions on the conducting wall at $y = -l$ are

$$E_x = E_{x1}^+ e^{ik_x x + ik_z z}(e^{ik_y y} - e^{-ik_y y}e^{-2ik_y l}) + \text{``}-,\text{''} \tag{25}$$

$$E_z = E_{z1}^+ e^{ik_x x + ik_z z}(e^{ik_y y} - e^{-ik_y y}e^{-2ik_y l}) + \text{``}-,\text{''} \tag{26}$$

$$B_x = \frac{ck_x}{\omega k_y}(E_{x1}^+ k_z - E_{z1}^+ k_x)e^{ik_x x + ik_z z}(e^{ik_y y} + e^{-ik_y y}e^{-2ik_y l}) + \text{``}-,\text{''} \tag{27}$$

$$B_z = \frac{ck_z}{\omega k_y}(E_{x1}^+ k_z - E_{z1}^+ k_x)e^{ik_x x + ik_z z}(e^{ik_y y} + e^{-ik_y y}e^{-2ik_y l}) + \text{``}-,\text{''} \tag{28}$$

where $k_y = i\sqrt{k_x^2 + k_z^2}$, and the coefficients $E_{x1}^{\pm}$ and $E_{z1}^{\pm}$ are arbitrary and different from those in the plasma.

We match the tangential components of electric and magnetic fields in plasma and in vacuum at the position $y_s = \frac{1}{2}(Ae^{ik_x x + ik_z z} + \text{c.c.})$. Then to the first order in A we find

$$E_{z1}^{+}(1-e^{-2ik_yl})-E_{z2}^{p+}=\tilde{E}_z^{+},$$

$$E_{x1}^{+}(1-e^{-2ik_yl})-E_{x1}^{p+}-\frac{k_xk_z}{k_z^2-\frac{\omega^2}{v_A^2}}E_{z2}^{p+}=\tilde{E}_x^{+},$$

$$\frac{k_z}{k_y}(1+e^{-2ik_yl})(E_{x1}^{+}k_z-E_{z1}^{+}k_x)+\frac{k_x^2+k_{y1}^2}{k_{y1}}E_{x1}^{p+}=\frac{\omega}{c}\tilde{B}_z^{+},$$

$$\frac{k_x}{k_y}(1+e^{-2ik_yl})(E_{x1}^{+}k_z-E_{z1}^{+}k_x)-\frac{k_xk_z}{k_{y1}}E_{x1}^{p+}+\frac{k_{y2}\frac{\omega^2}{v_A^2}}{k_z^2-\frac{\omega^2}{v_A^2}}E_{z2}^{p+}=-\frac{A}{2}\frac{\omega}{c}\beta k_{y0}+\frac{\omega}{c}\tilde{B}_x^{+}.$$

Similar equations are for the coefficients with "−." In the above equations we omitted superscripts "v" in the vacuum coefficients. Substituting E_{z1}^{+} and E_{x1}^{+} from the first two equations into the second two, we find

$$\left(\frac{k_z^2}{k_y}\triangle+\frac{k_x^2+k_{y1}^2}{k_{y1}}\right)E_{x1}^{p+}+\frac{k_z}{k_y}\triangle\frac{k_x\frac{\omega^2}{v_A^2}}{k_z^2-\frac{\omega^2}{v_A^2}}E_{z2}^{p+}=-\frac{k_z}{k_y}\triangle(k_z\tilde{E}_x^{+}-k_x\tilde{E}_z^{+})+\frac{\omega}{c}\tilde{B}_z^{+},$$

$$\left(\frac{k_xk_z}{k_y}\triangle-\frac{k_xk_z}{k_{y1}}\right)E_{x1}^{p+}+\left(\frac{k_{y2}\frac{\omega^2}{v_A^2}}{k_z^2-\frac{\omega^2}{v_A^2}}+\frac{k_x}{k_y}\triangle\frac{k_x\frac{\omega^2}{v_A^2}}{k_z^2-\frac{\omega^2}{v_A^2}}\right)E_{z2}^{p+}=-\frac{k_x}{k_y}\triangle(k_z\tilde{E}_x^{+}-k_x\tilde{E}_z^{+})-\frac{A}{2}\frac{\omega}{c}\beta k_{y0}+\frac{\omega}{c}\tilde{B}_x^{+}, \quad (29)$$

where

$$\triangle=\frac{1+e^{-2ik_yl}}{1-e^{-2ik_yl}}\approx-\frac{1}{|k_y|l},$$

since in the cases of interest $|k_y|l\ll 1$ (unstable modes have the wave length much longer than the width of the vacuum layer).

After solving Eqs. (29) one can find the field and current components in the plasma and then find the rf pressure distribution as

$$P=\int_{y_s}^{\infty}f_y dy, \quad (30)$$

where f is given by Eq. (16). For keeping the linear in A accuracy, the integration in Eq. (30) should be performed only accross the skin layer.

Careful analysis of Eqs. (29) and the subsequent calculation of the rf pressure shows that the rf pressure is stabilizing only in the limit when $k_z=0$. In this limit

$$P=P_0\left[1-2\frac{\frac{1}{l}\frac{4\pi\omega}{\eta c^2}+\frac{1}{l^2}\sqrt{\frac{4\pi\omega}{2\eta c^2}}}{\frac{4\pi\omega}{\eta c^2}+\frac{2}{l}\sqrt{\frac{4\pi\omega}{2\eta c^2}}+\frac{1}{l^2}}\times\frac{1}{2}(Ae^{ik_xx+ik_zz}+\text{c.c.})\right]. \quad (31)$$

In order to compare the results in this section with the ideal case of an unmagnetized perfectly conducting plasma, we provide here the equation for the pressure distribution for the simple model of Sec. II derived for the geometry of this section (Fig. 3),

$$P=\frac{E_A^2}{16\pi\sin^2\lambda}\left[1-2\frac{k_x^2}{|k_y|^2l}\frac{1}{2}(Ae^{ik_xx+ik_zz}+\text{c.c.})\right]. \quad (32)$$

The correction to the unperturbed pressure in the above equation is inversly proportional to the width of the vacuum layer l. The coefficient in front of the plasma surface displacement is negative because the y axis is directed towards the plasma now.

From Eq. (31) one can see that in the limit $k_z=0$ this equation reproduces the simplified profile of Eq. (32) ($|k_x|=|k_y|$ for $k_z=0$) when the width of the vacuum layer l is bigger than the skin depth.

Equation (31) is valid roughly when $k_z<\omega/v_A\sqrt{\triangle}$. This restriction on k_z is very limiting; it implies that the rf pressure in a magnetized plasma is stabilizing only for perturbations with $m=0$ (which is not the case of interest for the resistive wall modes).

When $k_z\gg\omega/v_A\sqrt{\triangle}$ (this condition includes $m=1$ perturbation in cylindrical geometry for typical plasmas), then the correction to the unperturbed pressure P_0 is roughly by a factor $\triangle$ smaller than that given by Eq. (32) [the relation of Eq. (15) is taken into account for this estimation]. In this case this correction is of the same order of magnitude as previously neglected terms, which means that it is zero within the accuracy of our calculations.

This significant change of the rf pressure distribution in the magnetized plasma is due to the change of the wave polarization in the magnetic field when $k_z\neq 0$ relative to the wave polarization in unmagnetized plasma. This results in erasing the perturbed rf pressure when the plasma boundary is perturbed, so that the rf pressure neither stabilizes nor destabilizes the plasma surface.

The unperturbed rf fields are driven with the wave number $k_z=0$, so that the Alfvén resonance ($k_z=\omega/v_A$) is not excited directly. One can assume that k_z of the perturbation is discrete (k_z corresponds to m in cylindrical geometry) and in general the condition for excitation of Alfvén resonance is not satisfied in the skin layer. Due to this we do not consider special case of the presence of Alfvén resonance in this model.

B. Magnetized plasma with a skin layer, $\omega_{\mathrm{rf}} \gtrsim \omega_{ci}$

In this section we briefly outline the calculations and results for the rf pressure distribution in the case of rf frequencies in the range $\omega \gtrsim \omega_{ci}$. For these frequencies the dielectric properties of a magnetized plasma are qualitatively different from those considered in the previous section, this might lead to a stabilizing rf presure distribution. A separate analysis is necessary to obtain the result in this case.

In this range of frequencies the propagation of em waves in plasma is properly described by a collisionless two fluid model. For the plasma parameters of the previous section, the skin depth now is a fraction of 1 cm. As before we neglect the plasma pressure contribution, then this model results in the description of the plasma dielectric properties by a cold plasma dielectric tensor.

We use the same coordinates as in the previous section, and again the driven rf electric field is parallel to the magnetic field $\mathbf{B}_0$.

If $\lambda = \omega l/c$, then the nonzero components of the unperturbed rf fields are in vacuum,

$$E_z = \left[-\sin(\omega y/c) + \frac{\omega}{\omega_{pe}} \cos(\omega y/c) \right] \frac{E_A}{\sin\lambda + \frac{\omega}{\omega_{pe}} \cos\lambda},$$

$$B_x = \left[\cos(\omega y/c) + \frac{\omega}{\omega_{pe}} \sin(\omega y/c) \right] \frac{iE_A}{\sin\lambda + \frac{\omega}{\omega_{pe}} \cos\lambda},$$

and in plasma,

$$E_{zp} = \frac{\omega}{\omega_{pe}} \frac{E_A}{\sin\lambda + \frac{\omega}{\omega_{pe}} \cos\lambda} e^{ik_{y0}y},$$

$$B_{xp} = \frac{iE_A}{\sin\lambda + \frac{\omega}{\omega_{pe}} \cos\lambda} e^{ik_{y0}y},$$

where $k_{y0} = i\omega_{pe}/c$.

Again we assume that the perturbation of the plasma–vacuum boundary is much smaller than the skin depth and that the condition of Eq. (13) is satisfied. We perform the linearization of the two fluid equations about the unperturbed oscillating quantities. Then we separate slow and fast time scales. We find that the time averaged force per unit volume acting on the plasma is

$$\mathbf{f} = \frac{1}{c} \langle j_{z0\mathrm{rf}} B_{x0\mathrm{rf}} \mathbf{e}_y + j_{z0\mathrm{rf}} \mathbf{e}_z \times \mathbf{B}_{\mathrm{rf}} + j_{z\mathrm{rf}} B_{x0\mathrm{rf}} \mathbf{e}_y \rangle. \tag{33}$$

In the derivation of this equation some smaller (for typical RFP plasmas) terms were neglected.

After tedious calculations one can find that the equations for the perturbed rf electric fields are similar to those of Eqs. (19) of the previous section,

$$-k^2 E_x + \frac{\omega^2}{c^2} \varepsilon_\perp E_x + \frac{\omega^2}{c^2} igE_y + k_x(k_x E_x + k_{y0} E_y + k_z E_z) = -\frac{i\omega k_{y0}\beta}{c} \frac{1}{2} \frac{B_x}{B_0},$$

$$-k^2 E_y + \frac{\omega^2}{c^2} \varepsilon_\perp E_y - \frac{\omega^2}{c^2} igE_x + k_{y0}(k_x E_x + k_{y0} E_y + k_z E_z) = -\frac{i\omega k_{y0}\beta}{c} \frac{1}{2} \frac{B_y}{B_0},$$

$$-k^2 E_z + \frac{\omega^2}{c^2} \varepsilon_\parallel E_z + k_z(k_x E_x + k_{y0} E_y + k_z E_z) = -\frac{ik_{y0}\beta k_z}{c} \frac{1}{2} v_{ez}, \tag{34}$$

where $\varepsilon_\perp$, $\varepsilon_\parallel$, g are the components of the cold plasma dielectric tensor (see, e.g., Ref. 11),

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{pmatrix},$$

and the quantities on the right-hand side of Eqs. (34) correspond to MHD perturbations.

As in the previous section we find the general solution of Eqs. (34) and match it with the solution in the vacuum layer applying similar boundary conditions on the plasma–vacuum surface. The resultant rf pressure distribution is similar to that obtained in the previous section. In the limit $k_z = 0$ the result of Eq. (32) is reproduced. As in the low frequency case this result is very restrictive. It is valid for $k_z < (\omega/c)\sqrt{|\varepsilon_\perp|}/\triangle$. For $k_z \gg (\omega/c)\sqrt{|\varepsilon_\perp|}/\triangle$ the correction to the unperturbed pressure is roughly a factor $\triangle$ smaller than that given by Eq. (32). Again this restricts the stabilizing effect only to $m=0$ perturbations.

C. Zero skin depth limit

We consider here the range of frequencies $\omega \gtrsim \omega_{ci}$. This range is investigated in the previous section with the assumption that the amplitude of plasma displacement A is smaller than the skin depth. Since the skin depth for these frequencies is relatively small, this assumption may be violated. In order to have a complete picture about the perturbation of rf pressure, we consider here the limit when the skin depth is much smaller than A. Because the expressions in this section are relatively simple, we can make a more definite conclusion on why the perturbed rf pressure is diminished for $k_z \neq 0$.

In this limit the electric field component parallel to the magnetic field in plasma is shielded, so that only one (propagating) branch is left for the rf fields in plasma. The approximate rf fields in plasma are

$$E_x = E_{x1}^{p+} e^{ik_{y1}y + ik_x x + ik_z z} + E_{x1}^{p-} e^{ik_{y1}y - ik_x x - ik_z z},$$

$$E_z = 0,$$

$$B_x = -\frac{ck_z}{\omega} \frac{k_x k_{y1} - \frac{\omega^2}{c^2} ig}{k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp} E_{x1}^{p+} e^{ik_{y1}y + ik_x x + ik_z z} + \text{``}-,\text{''}$$

$$B_z = -\frac{c}{\omega} \frac{k_{y1}\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right) + k_x \frac{\omega^2}{c^2} ig}{k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp} \times E_{x1}^{p+} e^{ik_{y1}y + ik_x x + ik_z z} + \text{``}-,\text{''}$$

the terms with the coefficients with "−" are obtained by changing $k_x \to -k_x$, $k_z \to -k_z$. The wave number k_{y1} in the above equations is the root of a cold plasma dispersion equation with fixed k_x, k_z, ω.

We now match the vacuum solution of Eqs. (25)–(28) with the plasma solution. The condition $\mathbf{E}\cdot\mathbf{B}=0$ on the plasma–vacuum boundary gives

$$E_{z1}^{+} = \frac{\omega}{c} \frac{\beta \frac{1}{2} A}{1 - e^{-2ik_y l}}, \tag{35}$$

$$E_{z1}^{-} = \frac{\omega}{c} \frac{\beta \frac{1}{2} A^*}{1 - e^{-2ik_y l}}. \tag{36}$$

From the condition $\mathbf{E}^v \cdot [\mathbf{n}\times\mathbf{B}] = \mathbf{E}^p \cdot [\mathbf{n}\times\mathbf{B}]$ we find

$$E_{x1}^{+}(1 - e^{-2ik_y l}) = E_{x1}^{p+}, \tag{37}$$

$$E_{x1}^{-}(1 - e^{-2ik_y l}) = E_{x1}^{p-}, \tag{38}$$

and from $\mathbf{B}^v \cdot \mathbf{B} = \mathbf{B}^p \cdot \mathbf{B}$ we have

$$i\beta \frac{1}{2} \frac{B_x}{B_0} + \frac{c}{\omega} \frac{k_z}{k_y} (E_{x1}^{+} k_z - E_{z1}^{+} k_x)(1 + e^{-2ik_y l}) = -\frac{c}{\omega} \frac{k_{y1}\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right) + k_x \frac{\omega^2}{c^2} ig}{k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp} E_{x1}^{p+}, \tag{39}$$

$$i\beta \frac{1}{2} \frac{B_x^*}{B_0} + \frac{c}{\omega} \frac{k_z}{k_y} (E_{x1}^{-} k_z - E_{z1}^{-} k_x)(1 + e^{-2ik_y l}) = -\frac{c}{\omega} \frac{k_{y1}\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right) - k_x \frac{\omega^2}{c^2} ig}{k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp} E_{x1}^{p-}. \tag{40}$$

We solve Eqs. (35)–(40) to find the perturbed B_x component in the vacuum layer given by Eq. (27). This component is used to find the perturbed rf pressure distribution on the plasma surface. After some calculations we find

$$B_x^{+} \propto \frac{k_x \beta \sigma^{+} A - i\beta k_z \frac{1}{2} \frac{B_x}{B_0}}{\frac{k_z^2}{k_y} + \frac{2\sigma^{+}}{\triangle}}, \tag{41}$$

where

$$\sigma^{+} = \frac{k_{y1}\left(k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp\right) + k_x \frac{\omega^2}{c^2} ig}{k_x^2 + k_z^2 - \frac{\omega^2}{c^2}\varepsilon_\perp}$$

and B_x on the right-hand side of Eq. (41) is the amplitude of the perturbed MHD magnetic field on the plasma boundary.

In the limit $k_z = 0$ the rf pressure perturbation found with B_x given by Eq. (41) is the same as in the simplified case of Eq. (32) ($l \gg c/\omega_{pe}$ is assumed). This limit is valid roughly for $k_z < (\omega/c)\sqrt{|\varepsilon_\perp|}/\triangle$. In the case when $k_z \gg (\omega/c)\sqrt{|\varepsilon_\perp|}/\triangle$, the first term in the denominator in Eq. (41) dominates and the pressure perturbation is roughly by a factor of $\triangle$ smaller than that given by Eq. (32) [here we assumed that in Eq. (41) $B_x \sim B_y = ik_z A B_0$].

From Eq. (41) one can see that the perturbed rf pressure is diminished when $k_z \neq 0$. This is because of the presence of the penetrating branch (with the wave number k_{y1}) in the magnetized plasma. Without the magnetic field, $\sigma^{+} \to \infty$ ($k_{y1} \to \infty$) and Eq. (41) results in the ideal pressure distribution of Eq. (32).

V. SUMMARY

We have investigated rf wave stabilization of MHD instability. This work contains two new elements. First, we examine stabilization of resistive wall instability, focusing on the example of an ideal kink instability in the reversed field pinch. The motivation is that if rf wave can be confined to a thin vacuum region, then the $j \times B$ force arising from the perturbed rf wave energy density provides a strong restoring force. Although the wall is resistive to the MHD instability, it traps the higher frequency rf waves. Second, we include the self-consistent effect of the MHD perturbation on the rf wave dynamics.

We find that the MHD perturbation of the plasma surface generates Alfvén disturbances—a variant of the shear Alfvén wave which evanesces into the plasma and a compressional wave which deeply penetrates. If we neglect the Alfvén wave generation, then the rf waves can stabilize the ideal kink resistive wall instability for rf wave magnetic fields that are much less than the equilibrium field. The rf waves simply evanesce with a resistive skin depth. However, the Alfvén waves which are unavoidably generated defeat the rf wave localization and the stabilization. This suppression of stabilization is found in three qualitatively different plasma cases, discussed in Sec. IV.

Thus, effects due to perturbation of rf intensity considered with the proper description of the plasma dielectric properties should be included in analysis of ponderomotive stabilization of external modes in fusion devices. A successful rf stabilization scheme would require selection of waves which, when perturbed by the MHD instability, only generate secondary waves which are confined to the plasma edge.

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