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Invited paper

Algebraically Simple Chaotic Flows

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Abstract[1]
It came as a surprise to most scientists when Lorenz in 1963 discovered chaos in a simple system of three autonomous ordinary differential equations with two quadratic nonlinearities. This paper reviews efforts over the subsequent years to discover even simpler examples of chaotic flows. There is reason to believe that the algebraically simplest examples of chaotic flows with quadratic and piecewise linear nonlinearities have now been identified. The properties of these and other simple systems will be described.

Keywords: ***chaos, flow, jerk, strange attractor, differential equations, fractal, bifurcation, circuits***

1. Introduction

Some aspects of chaos have been known for over a hundred years. Isaac Newton was said to get headaches thinking about the 3-body problem (Sun, Moon, and Earth). In 1890, King Oscar II of Sweden announced a prize for anyone who could solve the *n*-body problem and hence demonstrate stability of the solar system. The prize was awarded to Jules Henri Poincaré who showed that even the 3-body problem has no analytical solution [1, 2]. He went on to deduce many of the properties of chaotic systems including the sensitive dependence on initial conditions. With the successes of linear models in the sciences and the lack of powerful computers, the work of these early nonlinear dynamists went largely unnoticed and undeveloped for many decades.

In 1963, Lorenz published a seminal paper [3] in which he showed that chaos can occur in systems of autonomous (no explicit time dependence) ordinary differential equations (ODEs) with as few as three variables and two quadratic nonlinearities. For continuous flows, the Poincaré-Bendixson theorem [4] implies the necessity of three variables, and chaos requires at least one nonlinearity. More explicitly, the theorem states that the long-time

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limit of any "smooth" two-dimensional flow is either a fixed point or a periodic solution. With the growing availability of powerful computers, many other examples of chaos were subsequently discovered in algebraically simple ODEs. Yet the sufficient conditions for chaos in a system of ODEs remain unknown.

This paper will review early serendipitous and insightful discoveries as well as later extensive computer searches for the algebraically simplest chaotic flows. There are reasons to believe that the simplest such examples with quadratic and piecewise linear nonlinearities have now been identified. The properties of these systems will be described.

2. Lorenz and Rössler

The celebrated Lorenz equations are:

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz\end{aligned} \qquad (1)$$

where the dot denotes a first time derivative ($\dot{x} = dx/dt$, etc.). Note that there are seven terms on the right-hand side of these equations, two of which are nonlinear (*xz* and *xy*). Also note that there are three parameters, for which Lorenz found chaos with *ó* = 10, *r* = 28, and *b* = 8/3. The number of independent parameters is generally *d*+1 less than the number of terms for a *d*-dimensional system, since each of the variables (*x*, *y*, and z in this case) and time (*t*) can be arbitrarily rescaled. The Lorenz system has been extensively studied, and there is an entire book by Sparrow [5] devoted to it.

Although the Lorenz system is often taken as the prototypical chaotic flow, it is not the algebraically simplest such system. In 1976, Rössler [6] proposed the following equations:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + xz - cz\end{aligned} \qquad (2)$$

This example also has seven terms and three parameters, which Rössler took as *a* = *b* = 0.2 and *c* = 5.7, but only a single quadratic nonlinearity (*xz*).

As recently as 1993, Lorenz [7] wrote: "*One other study left me with mixed feelings. Otto Rössler of the University of Tübingen had formulated a system of three differential equations as a model of a chemical reaction. By this time, a number of systems of differential equations with chaotic solutions had been discovered, but I felt I still had the distinction of having found the simplest. Rössler changed things by coming along with an even simpler one. His record still stands*."

What Lorenz apparently did not realize was that Rössler himself had much earlier (in 1979) found an even simpler system [8] given by:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x \\ \dot{z} &= ay - ay^2 - bz\end{aligned} \qquad (3)$$

This system has only six terms, a single quadratic nonlinearity (y^2), and two parameters, giving chaos with *a* = 0.386 and *b* = 0.2. For some other values of the parameters, the dynamics are quasiperiodic, with a trajectory that lies on an invariant torus.

However, note that the simplicity of a system can be measured in various ways. Algebraic simplicity is one such way; topological simplicity is another. The Rössler attractor and most of the others in this paper are topologically simpler than the double-lobed attractor of Lorenz, but they are roughly equivalent in that they all tend to resemble the single folded-band structure produced by Eq. (2).

3. Computer Search

Also unaware of the simpler Rössler example, Sprott [9] embarked on an extensive search for autonomous three-dimensional chaotic systems with fewer than seven terms and a single quadratic nonlinearity and systems with fewer than six terms and two quadratic nonlinearities. The brute-force method [10, 11] involved the numerical solution of about 10^8 systems of autonomous ODEs with randomly chosen real coefficients and initial conditions. The criterion for chaos was the existence of a decidedly positive Lyapunov exponent [12].

He found fourteen algebraically distinct cases with six terms and one nonlinearity, and five cases with five terms and two nonlinearities. One case was volume-conserving (conservative), and all the others were volume-contracting (dissipative), implying the existence of a strange attractor. Sprott provided a table of the spectrum of Lyapunov exponents, the Kaplan-Yorke dimension [13], and the types and eigenvalues of the unstable fixed points for each of the nineteen cases. Interestingly, the Rössler example in Eq. (3) was not found, suggesting that even this extensive search was not exhaustive.

Subsequently, Hoover [14] pointed out that the conservative case A found by Sprott

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + yz \\ \dot{z} &= 1 - y^2 \end{aligned} \tag{4}$$

is a special case of the Nosé-Hoover thermostated dynamic system that had earlier been shown [15] to exhibit time-reversible Hamiltonian chaos. Note that this case in general needs an adjustable parameter, but it turns out that chaos occurs for all coefficients equal to unity, and so it is especially simple in that sense. None of the fourteen cases with a single quadratic nonlinearity share that property, although there are two other chaotic cases with all unity coefficients and two quadratic nonlinearities with strange attractors. Chaos is observed in Eq. (4) for only a small range of initial conditions, one choice of which is (x, y, z) = (0, 5, 0). The other eighteen chaotic cases were apparently previously unknown.

This search for algebraically simple chaotic systems was an outgrowth of earlier studies [16] in which Sprott showed that three-dimensional ODEs with quadratic nonlinearities and bounded solutions are chaotic for 0.38±0.02 percent of the cases with uniform randomly chosen coefficients. The probability of chaos increases approximately as the square root of the dimension up to at least d = 8 as shown in Fig. 1. Also shown in Fig. 1 is the probability of chaos in systems governed by difference equations (iterated maps), whose behavior is contrary to the case of ODEs (flows) for reasons that are only partly understood. Whereas ODEs with quadratic nonlinearities require three dimensions to exhibit chaos, iterated maps can be chaotic with only one dimension. Similar studies [17] with randomly connected, discrete-time, artificial neural networks with a hyperbolic tangent squashing function show that the probability of chaos is small at low dimension and increases to nearly 100% at a dimension of about 100. The relative rarity of chaos in low-dimensional ODE systems is the reason chaos went largely unnoticed for so long and why new examples of simple chaotic systems are still being discovered. In a related study [18] Sprott showed that the average correlation dimension of chaotic d-dimensional flows with quadratic nonlinearities and uniform randomly chosen coefficients is approximately $1.07d^{0.3}$ and the average largest Lyapunov exponent is approximately $1.15d^{-0.84}$

4. Jerk Functions

In response to Sprott's work, Gottlieb [19] pointed out that Eq. (4) could be recast in the explicit third-order form

$$\dddot{x} = -\dot{x}^3 + \ddot{x}(x + \ddot{x})/\dot{x} \tag{5}$$

which he called a "jerk function" since it involves a third derivative of *x*, which in a mechanical system is the rate of change of the acceleration, sometimes called the "jerk" [20]. It is known that any explicit ODE can be cast in the form of a system of coupled first-order ODEs, but the converse does not hold in general. Even if one can reduce the dynamical system to a jerk form for each of the phase space variables, the resulting differential equations may look quite different. Gottlieb asked the provocative question "What is the simplest jerk function that gives chaos?"

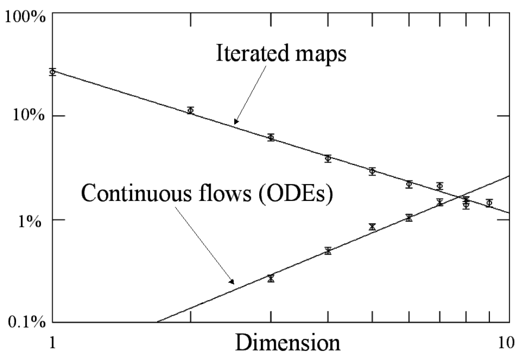

Figure 1. Probability of chaos for maps and flows of various dimensions.

One response was provided by Linz [21] who showed that the original Rössler model, the Lorenz model, and Sprott's case R can be reduced to jerk forms, albeit of very different complexity. The Rössler model in slightly modified form can be written as

$$\begin{aligned}&\dddot{y} + [c - \varepsilon + \varepsilon y - \dot{y}]\ddot{y} \\ &+ [1 - \varepsilon c - (1+\varepsilon^2)y + \varepsilon\dot{y}]\dot{y} \\ &+ (\varepsilon y + c)y + \varepsilon = 0\end{aligned} \qquad (6)$$

where *å* (= *a* = *b*) = 0.2 and *c* = 5.7 gives chaos. Note that the jerk form of the Rössler equation is a rather complicated quadratic polynomial with 10 terms. Figure 2 shows that the attractor in the $\dot{y} - y$ phase space is the familiar folded band.

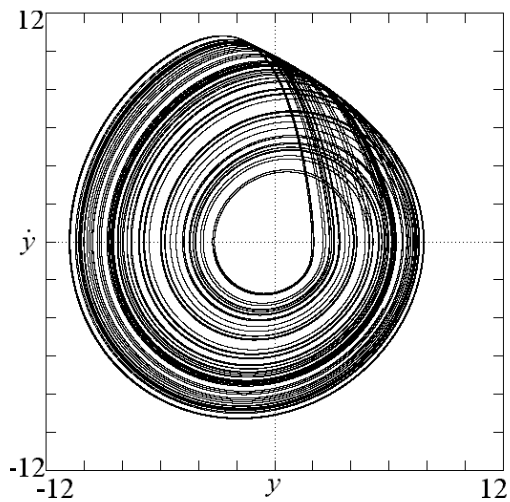

Figure 2. Phase-space plot of the jerk representation of the Rössler attractor from Eq. (6) with *å* = 0.2 and *c* = 5.7.

The Lorenz model in Eq. (1) can be written as

$$\begin{aligned}&\dddot{x} + [1 + \sigma + b - \dot{x}/x]\ddot{x} \\ &+ [b(1+\sigma+x^2) - (1+\sigma)\dot{x}/x]\dot{x} \\ &- b\sigma(r - 1 - x^2)x = 0\end{aligned} \qquad (7)$$

The jerk form of the Lorenz equation is not a polynomial since it contains terms proportional to $\dot{x}/x$ as is typical of dynamical systems with multiple nonlinearities. Its jerk form contains eight terms. The phase-space plot of Eq. (7) in Fig. 3 shows the familiar double-lobed attractor.

Linz showed that Sprott's case R can be written as a polynomial with only five terms and a single quadratic nonlinearity

$$\dddot{x} + \ddot{x} - x\dot{x} + ax + b = 0 \qquad (8)$$

with chaos for a = 0.9 and b = 0.4. Its attractor as shown in Fig. 4 is a folded band similar to the Rössler attractor.

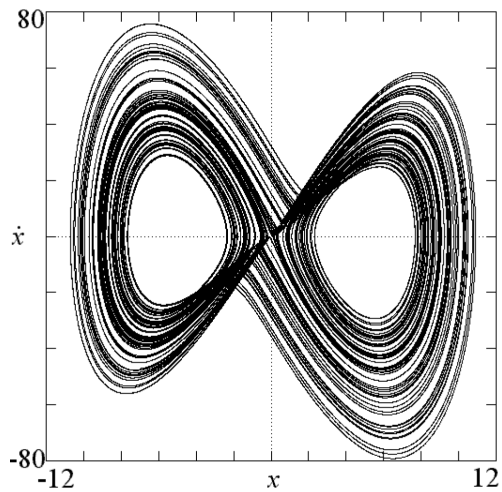

Figure 3. Phase-space plot of the jerk representation of the Lorenz attractor from Eq. (7) with ó = 10, r = 28, and b = 8/3.

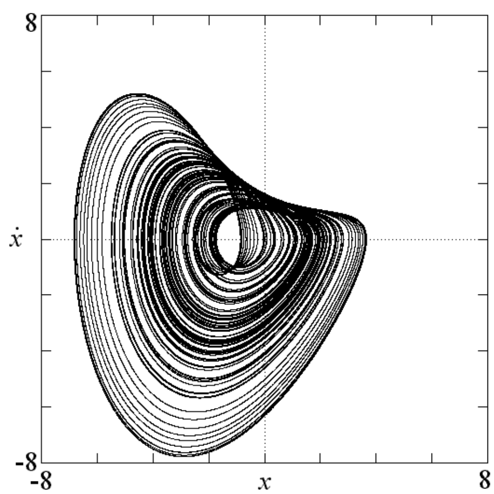

Figure 4. Attractor for Sprott's Case R from Eq. (8) with a = 0.9 and b = 0.4.

Clearly, the Lorenz and Rössler models are not candidates for Gottlieb's simplest jerk function that gives chaos, and Sprott's models demonstrate the existence of much simpler examples.

Meanwhile, Sprott also took up Gottlieb's challenge and embarked on an extensive numerical search for chaos in systems of the explicit form $\dddot{x} = J(\ddot{x}, \dot{x}, x)$, where the (jerk) function J is a simple quadratic or cubic polynomial. He found a variety of cases [22], including two with three terms and two quadratic nonlinearities in their jerk function,

$$\dddot{x} + ax\ddot{x} - \dot{x}^2 + x = 0 \tag{9}$$

with a = 0.645 and

$$\dddot{x} + ax\ddot{x} - x\dot{x} + x = 0 \tag{10}$$

with a = **-** 0.113, and a particularly simple case with three terms and a single quadratic nonlinearity [23],

$$\dddot{x} + a\ddot{x} \pm \dot{x}^2 + x = 0 \tag{11}$$

with a = 2.017. Its attractor is shown in Fig. 5. For this value of a, the Lyapunov exponents (base-e) are (0.0550, 0, -2.0720) and the Kaplan-Yorke dimension is D_{KY} = 2.0265. He also found systems of the form

$$\dddot{x} + a\ddot{x} + \dot{x} = G(x) \tag{12}$$

where $G(x)$ is a second-degree (or higher) polynomial such as $x^2 - b$ or $x(x - b)$.

It is interesting to rewrite Eq. (11) as a dynamical system in the variables x, y, and z:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az \pm y^2 - x \end{aligned} \tag{13}$$

In this form, it is apparent that it has two fewer terms than the Lorenz or Rössler models and only a single quadratic nonlinearity (y^2). As a consequence, it is characterised by a single parameter (a). It is unlikely that a simpler quadratic form exists because it would have no adjustable parameters. The number of possibilities is quite small, and a systematic numerical check revealed that none of them exhibits chaos.

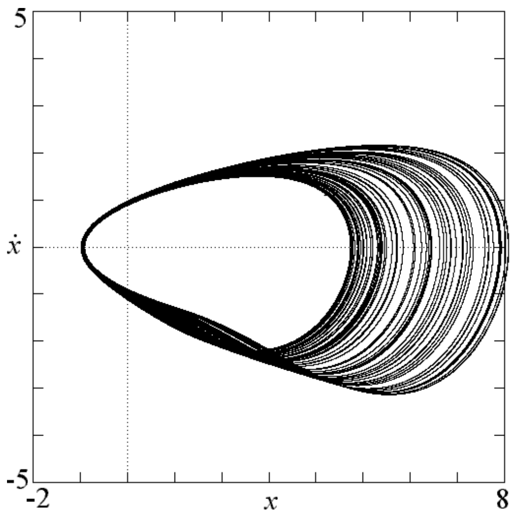

Figure 5. Attractor for the simplest chaotic flow with a quadratic nonlinearity from Eq. (11) with $a = 2.017$.

Equation (11) is simpler than any previously discovered case, both in its jerk representation and in its representation as a dynamical system. It was apparently overlooked in earlier searches because the range of a over which chaos occurs is quite narrow ($2.0168\ldots < a < 2.0577\ldots$). It also has a relatively small basin of attraction, so that initial conditions must be chosen carefully. One choice of initial conditions that lies in the basin of attraction is $(x, y, z) = (0, 0, \pm 1)$, where the sign is chosen according to the sign of the y^2 term in Eq. (13).

There is an alternate form for the simplest quadratic jerk function that can be written as

$$\dddot{x} + a\ddot{x} \pm x\dot{x} + x = 0 \qquad (14)$$

but this case is equivalent to Eq. (11) to within a constant as can be seen by differentiating Eq. (11) with respect to time and then renaming the variable $\dot{x}$ to x.

These systems, and most of the others in this paper, share a common route to chaos. The control parameter a can be considered a damping rate for the nonlinear oscillator. For large values of a, there are one or more stable equilibrium points. As a decreases, a Hopf bifurcation occurs in which the equilibrium becomes unstable, and a stable limit cycle is born. The limit cycle grows in size until it bifurcates into a more complicated limit cycle with two loops, which then bifurcates into four loops, and so forth, in a sequence of period doublings until chaos finally onsets. A further decrease in a causes the chaotic attractor to grow in size, passing through infinitely many periodic windows, and finally becoming unbounded when the attractor grows to touch the boundary of its basin of attraction (a crisis). A bifurcation diagram for Eq. (11), which is typical, is shown in Fig. 6. In this figure, the local maxima of x are plotted as the damping a is gradually decreased. Note that the scales are plotted backwards to emphasise the similarity to the logistic map, $x_{n+1} = Ax_n(1 - x_n)$ [24]. Indeed, a plot the maximum of x versus the previous maximum shows an approximate parabolic dependence, albeit with a very small-scale fractal structure. No cases were found with a toroidal attractor. Apparently, three-dimensional systems with a single quadratic nonlinearity cannot produce toroidal attractors.

Sprott also found a variety of chaotic jerk functions with cubic nonlinearities, the simplest of which have three terms and two nonlinearities. Two examples are

$$\dddot{x} + \ddot{x}^3 + x^2\dot{x} + ax = 0 \qquad (15)$$

with $a = 0.25$ and

$$\dddot{x} + a\ddot{x} - x\dot{x}^2 + x^3 = 0 \qquad (16)$$

with a = 3.6. He also found cases with four terms and one nonlinearity, including the old (circa 1966), but little known, Moore-Spiegel oscillator [25]

$$\dddot{x} + \ddot{x} + (T - R + Rx^2)\dot{x} + Tx = 0 \tag{17}$$

which models the inviscid convection of a rotating fluid, where T is analogous to the Prandtl number times the Taylor number and R is analogous to the Prandtl number times the Rayleigh number. Moore and Spiegel reported "aperiodic behaviour" for T = 6 and R = 20; the term "chaos" was not coined until 1975 [26].

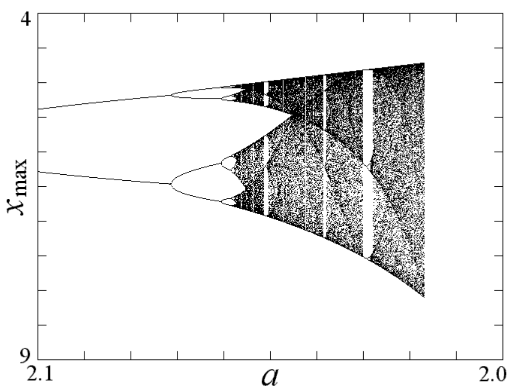

Figure 6. Bifurcation diagram for Eq. (11) as the damping is reduced.

A phase-space plot of the Moore-Spiegel attractor for these parameters is shown in Fig. 7. The regions in R-T space over which chaos occurs are shown in Fig 8, in which chaos is assumed to exist if the largest Lyapunov exponent exceeds 0.005 after 4×10^5 fourth-order Runge-Kutta iterations with a step size of 0.05. A more general equation of the same form as Eq. (17) was derived by Auvergne and Baglin [27] to model the motion of the ionisation zone of a star, and they also reported "irregular behaviour" and a broad power spectrum.

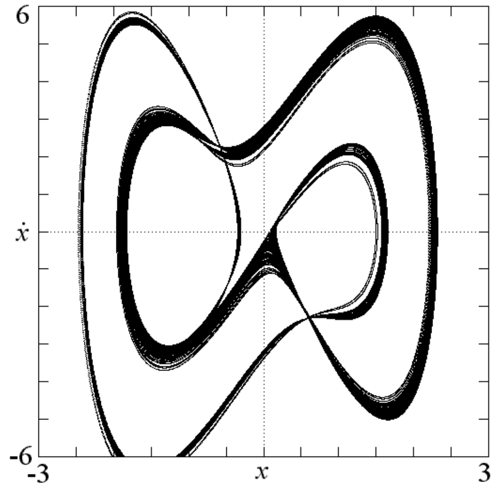

Figure 7. Attractor for the Moore-Spiegel oscillator in Eq. (17) with T = 6 and R = 20.

Sprott did not find dissipative chaotic jerk functions with fewer than four terms and a single cubic nonlinearity. The absence of chaos in Eq. (11) with a cubic instead of quadratic nonlinearity is curious since it contradicts the conventional wisdom that increasing the nonlinearity enhances the likelihood of chaos. It is evident that a certain amount of nonlinearity is required for chaos, but more is not necessarily better.

Recently, Malasoma [28] joined the search for simple chaotic jerk functions and found a cubic case as simple as Eq. (11) but of a different form

$$\dddot{x} + a\ddot{x} - x\dot{x}^2 + x = 0 \tag{18}$$

which exhibits chaos for a = 2.05. Its attractor is shown in Fig 9. For this value of a, the Lyapunov exponents (base-e) are (0.0541, 0, -2.1041), and the Kaplan-Yorke dimension is D_{KY} = 2.0257. This case follows the usual period-doubling route to chaos, culminating in a boundary crisis and unbounded solutions as a is lowered. The range of a over which chaos occurs is very narrow, 2.0278… < a < 2.0840…, which probably explains why it was missed in earlier numerical searches. There is also a second extraordinarily small window of chaos for 0.0753514… < a < 0.0753624…, which is five thousand times smaller than the previous case. Malasoma points out that this system is invariant under the parity transformation $x \to -x$ and speculates that this system is the simplest such example.

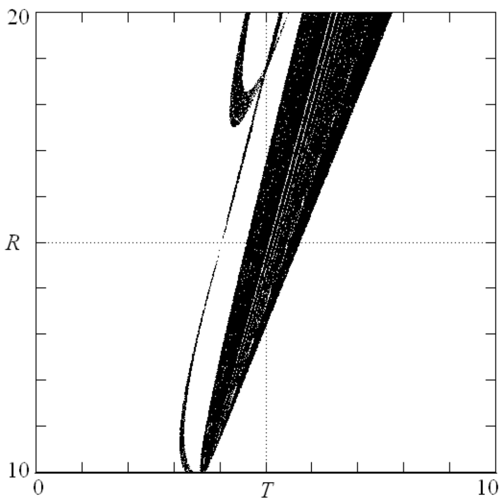

Figure 8. Regions of *R*-*T* space over which chaos occurs for the Moore-Spiegel oscillator in Eq. (17).

Both Linz and Sprott pointed out that if the jerk function is considered the time derivative of an acceleration of a particle of mass m, Newton's second law implies a force F whose time derivative is $dF/dt = mJ$. If the force has an explicit dependence on only $\dot{x}$, x, and time, it is considered to be "Newtonian jerky". The condition for $F = F(\dot{x}, x, t)$ is that J depends only linearly on $\ddot{x}$. In such a case the force in general includes a memory term of the form

$$M = \int^{t} G(x(\tau))d\tau \qquad (19)$$

that depends on the dynamical history of the motion.

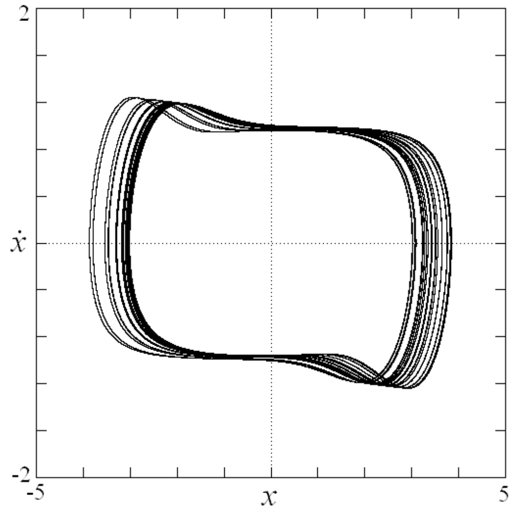

Figure 9. Attractor for the simple cubic flow in Eq. (18) with a = 2.05.

The jerk papers by Linz and by Sprott appeared in the same issue of the American Journal of Physics and prompted von Baeyer [29] to comment: "*The articles with those funny titles are not only perfectly serious, but they also illustrate in a particularly vivid way the revolution that is transforming the ancient study of mechanics into a new science—one that is not just narrowly concerned with the motion of physical bodies, but that deals with changes of all kinds.*" He goes on to say that the method of searching for chaos in a large class of systems "*is not just empty mathematical formalism. Rather it illustrates the arrival of a new level of abstraction in physical science… At that higher level of abstraction, dynamics has returned to the classical Aristotelian goal of trying to understand all change.*"

Eichhorn, Linz and Hänggi [30] summarized the situation with quadratic jerk functions that exhibit chaos. They used the method of comprehensive Gröbner bases [31] to show that all the previously mentioned chaotic flows with a single quadratic nonlinearity and some of those with multiple nonlinearities can be organized into a hierarchy of quadratic jerk equations with increasingly many terms. A slightly modified version of their results is shown in Table 1 with simplified parameters that produce chaos. Case JD_0 (not included by Eichhorn, et. al.) is a special case of JD_1 in which the constant term is zero. Such a categorization provides a simple means to compare the functional complexity of different systems and demonstrates the equivalence of cases not otherwise apparent. In a subsequent paper [32], the authors examined the simple cases JD_1 and JD_2 in more detail and identified the regions of parameter space over which they exhibit chaos, albeit with different forms chosen so that the locations of the fixed points are independent of the parameter values. In particular, JD_1 becomes

$$\dddot{x} = -\ddot{x} - a\dot{x} - bx + x\dot{x} \tag{20}$$

and JD_2 becomes

$$\dddot{x} = -\ddot{x} - a\dot{x} - b(x^2 - 1/4) \tag{21}$$

They also derived criteria for functional forms of the jerk function that cannot exhibit chaos. In particular, they showed that a jerk equation whose integral can be written as

$$\ddot{x} + \Omega(x, \dot{x}) = \int^t G(x(\tau))d\tau \tag{22}$$

with G being either a positive or a negative semidefinite function for all x cannot exhibit chaos. Moreover, if G is of the form $G(x) = \tilde{G}(x) + c$ with a positive (negative) constant c and a positive (negative) semidefinite function $\tilde{G}$, the dynamics eventually diverge for all initial values, except for those that coincide with fixed points.

Table 1. Classification of simple chaotic polynomial jerk systems

Model	Equation	Parameters
JD_0	$\dddot{x} = a\ddot{x} + \dot{x}^2 - x$	a = -2.017
JD_1	$\dddot{x} = a\ddot{x} + bx + x\dot{x} - 1$	a = -1.8 b = -2
JD_2	$\dddot{x} = a\ddot{x} + b\dot{x} + x^2 - 1$	a = - 0.5 b= - 1.9
JD_3	$\dddot{x} = a\ddot{x} + b\dot{x} + cx^2 + x\dot{x} - 1$	a = - 0.6 b = -3 c = 5
JD_4	$\dddot{x} = a\ddot{x} + b\dot{x} + cx^2 + x\ddot{x} - 1$	a = - 0.6 b = -2 c = 3
JD_5	$\dddot{x} = a\dot{x} + bx^2 + \dot{x}^2 - x\ddot{x}$	a = 0.5 b = -1
JD_6	$\dddot{x} = a\ddot{x} + b\dot{x} + cx^2 + d\dot{x}^2 + x\ddot{x} - 1$	a = -1 b = -1 c = 2 d = 2
JD_7	$\dddot{x} = a\ddot{x} + b\dot{x} + cx^2 + d\dot{x}^2 + ex\dot{x} + x\ddot{x} - 1$	a = -1 b = 1 c = 2 d = -3 e = 1

Concurrently, Fu and Heidel [33], with a technical correction by Gascon [34], proved that all three-dimensional, dissipative, dynamical systems with quadratic nonlinearities and fewer than five terms cannot exhibit chaos. They subsequently extended their results [35] to include conservative cases. More precisely, they rigorously proved their results for almost all conservative cases. The lone exception, $\dddot{z} = \dot{z}^2 - z^2$, appears numerically to have only periodic and unbounded solutions. Their work lends credence to the claim that Eq. (11) is the simplest chaotic flow with a quadratic nonlinearity.

Working independently, Thomas [36, 37] considered the terms in the Jacobian matrix of the system as feedback loops from which he was able to deduce candidate chaotic systems and the required signs of the coefficients. One such system is

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= x^2 - cz \end{aligned} \qquad (23)$$

with a = 0.385 and c = 2. This system can be reduced to the jerk form

$$\begin{aligned} &\dddot{x} + (c-a)\ddot{x} + (1-ac)\dot{x} \\ &+ 2x\dot{x} - ax^2 + cx = 0 \end{aligned} \qquad (24)$$

which by a transformation of variables is equivalent to case JD_3 in Table 1. The x^2 term in Eq. (23) can be replaced with other nonlinearities, including x^3, tanh(x), sin(x), and sgn(x), yielding chaos for appropriate parameter values. These cases have multiple equilibrium points.

Thomas also proposed a case with a single equilibrium (at the origin) given by

$$\begin{aligned} \dot{x} &= ax - y - z \\ \dot{y} &= x \\ \dot{z} &= x^2 - cz \end{aligned} \qquad (25)$$

with a = 0.25 and c = 2. Its jerk form is

$$\begin{aligned} &\dddot{x} + (c-a)\ddot{x} + (1-ac)\dot{x} \\ &+ 2x\dot{x} + cx = 0 \end{aligned} \qquad (26)$$

which is a generalization of Eq. (14). By a linear transformation and rescaling, Eq. (26) is equivalent to case JD_1 in Table 1. The x^2 term in Eq. (25) can be replaced with x^3, yielding chaos for a = 3.3 and c = 4, and a jerk function similar to but slightly more complicated than Eq. (18).

Thomas also suggested symmetric equations of the form

$$\begin{aligned} \dot{x} &= -ax + f(y) \\ \dot{y} &= -ay + f(z) \\ \dot{z} &= -az + f(x) \end{aligned} \qquad (27)$$

for which he found chaotic attractors for cubic polynomial and sinusoidal functions *f*. For example, a = 0.18 and $f(x)$ = sin(x) gives chaos as shown in Fig. 10. With $f(x)$ = sin(x), he also observed chaos in the conservative limit of a = 0. This particularly simple and elegant example has a trajectory that percolates chaotically within the infinite three-dimensional lattice of unstable steady states. Thomas calls this "labyrinth chaos." Unfortunately, its jerk representation is complicated because of the three nonlinearities. Equation (27) is a special case of the more general, cyclically symmetric system

$$\begin{aligned} \dot{x} &= f(x,y,z) \\ \dot{y} &= f(y,z,x) \\ \dot{z} &= f(z,x,y) \end{aligned} \qquad (28)$$

which has chaotic solutions for many choices of the nonlinear function *f*.

A convenient feature of chaotic jerk equations is that all three of the Lyapunov exponents can be determined from a numerical calculation of only the largest exponent. This exponent ($\ddot{e}_1$) must be positive for chaos, and there must be one zero exponent ($\ddot{e}_2$) corresponding to the direction of the flow. However, for a bounded system, the sum of the exponents $\ddot{e}_1 + \ddot{e}_2 + \ddot{e}_3$ is the rate of volume expansion averaged along the orbit and must be negative or zero and given by $\partial J/\partial \ddot{x}$, which is equal to a for the Newtonian jerky cases JD_0 though JD_3 in Table 1. Hence the negative exponent is easily found from $\ddot{e}_3 = a - \ddot{e}_1$. The Kaplan-Yorke dimension then follows from $D_{KY} = 2 - \ddot{e}_1/\ddot{e}_3$.

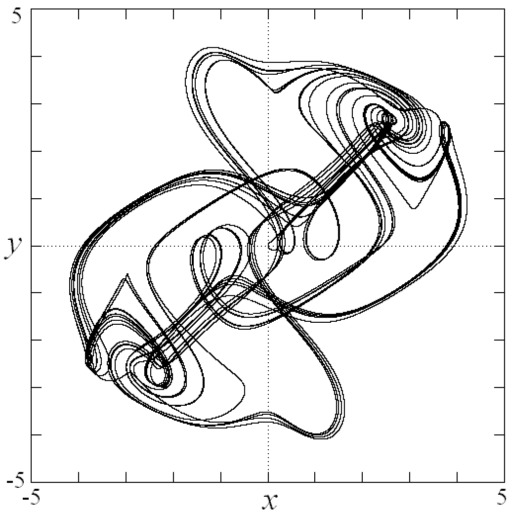

Figure 10. Attractor for the symmetric system of Thomas in Eq. (27) with a = 0.18 and $f(x) = \sin(x)$.

Cases whose exponents sum to zero are conservative, and their orbits fill a three-dimensional volume. Cases whose exponents sum to a negative value are dissipative, and they have a strange attractor with a dimension between 2 and 3. In either case, initial conditions must be chosen appropriately to ensure that they are in the basin of attraction for the dissipative systems and in the stochastic sea for the conservative systems. The emphasis here is on dissipative systems since they are more mathematically tractable and are better models of most natural systems.

5. Piecewise Linear Jerk Functions

Having found what appears to be the simplest jerk function with a quadratic nonlinearity that leads to chaos, it is natural to ask whether the nonlinearity can be weakened. In particular, the $\dot{x}^2$ in Eq. (11) might be replaced with $|\dot{x}|$. A numerical search did not reveal any such chaotic solutions.

However, one can formulate the question differently. Consider the system,

$$\dddot{x} + a\ddot{x} \pm |\dot{x}|^b + x = 0 \quad (29)$$

which is equivalent to Eq. (11) when $b = 2$. For what values of a and b does this system exhibit chaos? Figure 11 shows the result of a numerical search in which chaos is assumed to exist if the largest Lyapunov exponent exceeds 0.005 after 4×10^5 fourth-order Runge-Kutta iterations with a step size of 0.05. There are indeed regions of chaos for 1 < b 2 as well as for larger values of b. However, there do not appear to be chaotic solutions for $b = 3$ as mentioned earlier. The spiral structure of the chaotic region in a-b space begs for an explanation. It appears that the region of chaos does not extend down to $b = 1$, but this is because systems with $|x| + |y| + |z| > 10^6$ have been discarded since they are considered unbounded. In fact, the attractor size is found to increase approximately as $\exp[1/(b\text{-}1)]$ for $1 < b < 2$.

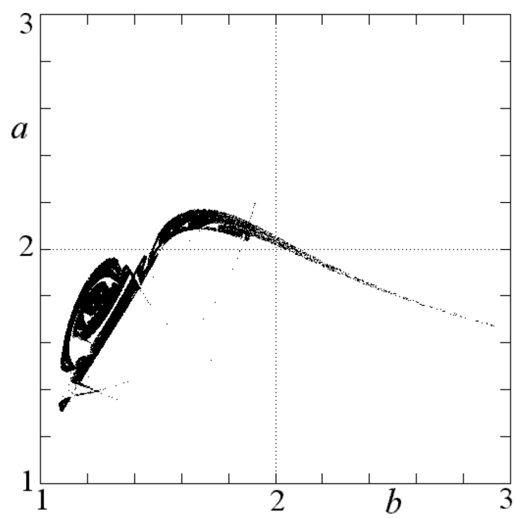

Figure 11. Regions of a-b space for which chaos occurs in Eq. (29).

In an extensive numerical search for the algebraically simplest dissipative chaotic flow with an absolute-value nonlinearity, Linz and Sprott [38] discovered the case

$$\dddot{x} + a\ddot{x} + b\dot{x} - |x| + 1 = 0 \qquad (30)$$

which exhibits chaos for a = 0.6 and b = 1. Chaos also occurs if the signs of both the last two terms are reversed, with an attractor that is a mirror image of the original about the x=0 plane. The attractor for this case as shown in Fig. 12 resembles the folded-band structure of the Rössler attractor.

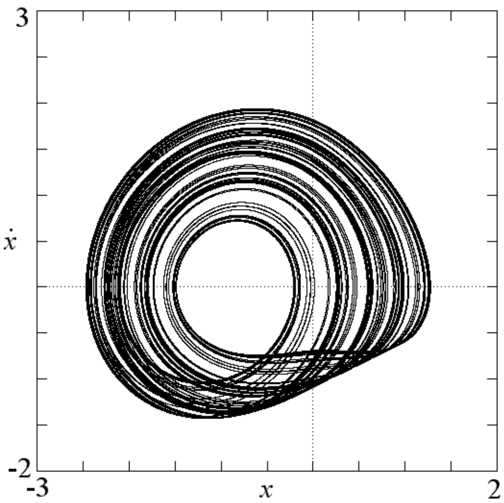

Figure 12. Attractor for the simplest chaotic system with an absolute-value nonlinearity from Eq. (30) with a = 0.6 and b = 1.

For these parameters, the Lyapunov exponents (base-e) are (0.035, 0, - 635), and the Kaplan-Yorke dimension is D_{KY} = 2.055. The abrupt change in direction of the flow at x = 0 is not evident in the figure because the discontinuity occurs only in the fourth time derivative of x.

The constant 1 in Eq. (30) affects only the size of the attractor. Chaos exists for arbitrarily small values of this constant, but the attractor and its basin of attraction shrink proportionally. This system exhibits a period-doubling route to chaos as shown in Fig. 13 and otherwise resembles the quadratic chaotic jerk functions previously described. This example relates to the quadratic flows as the tent map does to the logistic map. Linz and Sprott claim it is the most elementary piecewise linear chaotic flow and point out that the piecewise linear nature of the nonlinearity allows for an analytic solution to Eq. (30) by solving two linear equations and matching the boundary conditions at x = 0. Figure 14 shows the regions in a-b space for which chaos occurs in Eq. (30), in which chaos is assumed to exist if the largest Lyapunov exponent exceeds 0.005 after 4×10^5 fourth-order Runge-Kutta iterations with a step size of 0.05.

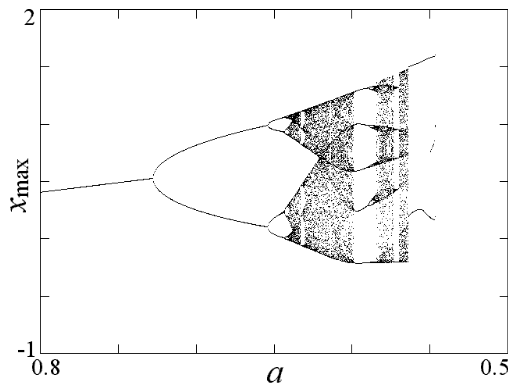

Figure 13. Bifurcation diagram for Eq. (30) with b = 1 as the damping is reduced.

More recently, Linz [39] has proved that chaos cannot exist in Eq. (30) if any of the terms are set to zero. He also notes that chaos is possible if the $|x|$ term in Eq. (30) is replaced with $|x^n|$, $|x|^n$, or x^{2n}, with n a positive integer, or more generally with any inversion symmetric function $f(x) = f(-x)$ that is either positive or negative for all x. Numerical experiments indicate that chaotic solutions with $f(x) = |x|^n$ exist for all nonzero n, including non-integer and negative values.

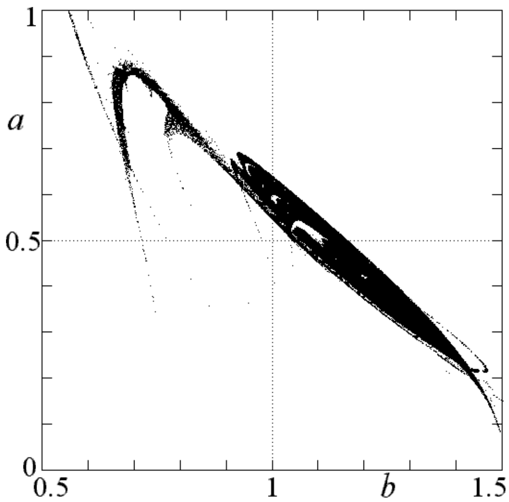

Figure 14. Regions of *a*-*b* space for which chaos occurs in Eq. (30).

In a recent paper, Fischer, Weiler, Fröhlich, and Rössler [40] propose a piecewise linear system whose jerk representation can be written as

$$\dddot{x} + a\ddot{x} + b\dot{x} + x - |x| + c = 0 \qquad (31)$$

which exhibits chaos for $a = c = 0.1$ and $b = 0.3$. It is intended to model an experimental example of chaos discovered in an electronic circuit [41]. They claim their system is "a maximally simple 3-variable ODE with a single letter-V shaped nonlinearity". Unfortunately, it has one more term and one more parameter than the case in Eq. (30), and thus it is not "maximally simple" in this sense, although the term x - $|x|$ can be written as 2 min(x, 0).

Equation (30) and (31) are special cases of Eq. (12) in which $G(x)$ is a general nonlinear function with the properties described below. Integrating each term in Eq. (12) reveals that this system is a damped harmonic oscillator driven by a nonlinear memory term that involves the integral of $G(x)$ as shown in Eq. (22). Such an equation often arises in the feedback control of an oscillator in which the experimentally accessible variable is a transformed and integrated version of the fundamental dynamical variable. Despite its importance and the richness of its dynamics, this system has been relatively little studied. Coullet, Tresser, and Arneodo observed chaos in numerical simulations with a cubic nonlinearity [42] of the form $G(x) = bx(1 - x^2)$ with $a = 0.1$ and $b = 0.44$ and with a special piecewise linear [43, 44] form

$$G(x) = \begin{cases} -bx - b - c & \text{if } x \le -1 \\ cx & \text{if } |x| \le 1 \\ -bx + b + c & \text{if } x \ge 1 \end{cases} \qquad (32)$$

with $a = 0.1$, $b = 0.2061612$, and $c = 0.2171604$ that models a cubic nonlinearity and satisfies the Sil'nikov conditions for a homoclinic orbit [45-47]. Thus it is one case for which chaos can be rigorously proved rather than numerically indicated.

It does not appear generally known that chaos accompanies many functions $G(x)$, some examples of which are listed in Table 2. These systems are elementary, both in the sense of having the algebraically simplest autonomous ODE and in the form of the nonlinearity. The table lists typical values of b that give chaos for arbitrary values of c with $a = 0.6$, along with the numerically calculated largest Lyapunov exponents (LE) in base-e. The constant c is arbitrary and only affects the size of the attractor.

For bounded solutions, $G(x)$ must average to zero along the orbit, which means that any continuous $G(x)$ must have at least one zero at $x = x^*$. The stability of the fixed point at (x^*, 0, 0) is determined by the solutions of the eigenvalue equation $\lambda^3 + a\lambda^2 + \lambda - G' = 0$, where $G' = dG/dx$, evaluated at $x = x^*$. This point is locally stable for $-a$ G' 0 and undergoes a Hopf bifurcation at $G' = -a$, where ë = ± i. Thus, one would expect chaotic systems of this form to require nonlinearity with either a positive slope at its zero crossing, or a large negative slope. Systems with $G' > 0$ apparently require at least two fixed points for chaos, but systems with $G' < -a$ only need one. All the cases in Table 2 have these properties. A scaling that preserves G' and the shape of $G(x)$ only affect the size of the attractor.

Table 2. Some simple functions $G(x)$ that produce chaos in Eq. (12) with a = 0.6.

$G(x)$	b	LE
$\pm(b\,\lvert x \rvert - c)$	1.0	0.036
$-b\max(x,0)+c$	6.0	0.093
$bx - c\,\mathrm{sgn}(x)$	1.2	0.657
$-bx + c\,\mathrm{sgn}(x)$	1.2	0.162
$\pm b(x^2/c - c)$	0.58	0.073
$bx(x^2/c - 1)$	1.6	0.103
$-bx(x^2/c - 1)$	0.9	0.126
$-b[x - 2\tanh(cx)/c]$	2.2	0.221
$\pm b\sin(cx)/c$	2.7	0.069
$\pm b\cos(cx)/c$	2.7	0.069

It is interesting to identify the maximally chaotic piecewise linear system. Of the cases in Table 2, the largest Lyapunov exponents occur for systems with $G(x) = bx - c\,\mathrm{sgn}(x)$. Using a variant of simulated annealing [48], the parameters a and b were adjusted to maximise the Lyapunov exponent. The result was a = 0.55 and b = 2.84, for which the Lyapunov exponents (base-e) are (1.055, 0, -1.655), giving an attractor with a Kaplan-Yorke dimension of D_{KY} = 2.637. The attractor as shown in Fig. 15 is contained within an extremely thin torus that nearly touches the boundary of its small basin of attraction so that initial conditions must be chosen carefully to produce bounded solutions. Initial conditions that suffice are (x, y, z) = (0.03, -0.33, -0.3).

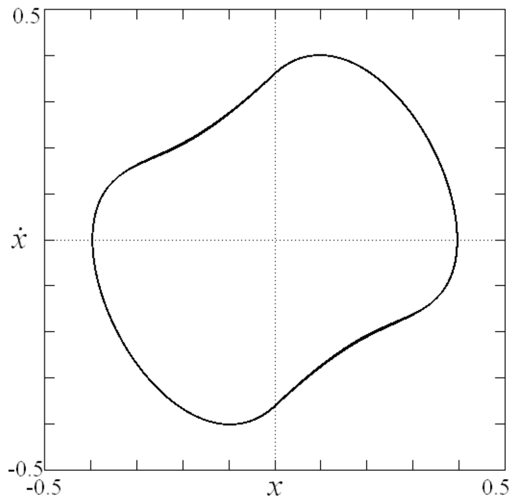

Figure 15. Attractor for the maximally chaotic system given by Eq. (12) with a = 0.55 and $G(x) = 2.84x - \mathrm{sgn}(x)$.

It is also interesting to identify the least nonlinear form of $G(x)$ for which chaos occurs, which we take to mean the two-part piecewise linear function with the smallest bend at the knee, è. Of the cases in Table 2, this condition occurs for $G(x) = \pm(b|x| - c)$ with a = 0.025 and b = 0.468, for which è (= $2\tan^{-1}b$) is about 50.2. The basin of attraction is very small, and the chaotic attractor coexists with a nearby limit cycle. Initial conditions that suffice are (x, y, z) = (0.9, 0, 0). Its attractor is shown in Fig. 16.

The chaotic cases described above by no means exhaust the list of simple jerk functions with chaotic solutions. In an extensive search for chaos in equations of the form

$$\dddot{x} = a\ddot{x} + b\varphi(\ddot{x}) + c\dot{x} + d\varphi(\dot{x}) + ex + f\varphi(x) + g \quad (33)$$

where $j\,(x)$ is one of a variety of simple nonlinear functions, several dozen algebraically distinct cases were found with three terms on the right-hand side, and several hundred cases were found with four terms on the right-hand side. Cases whose coefficients have different signs are considered distinct, but not cases in which only the values differ. Table 3 lists a small selection of those cases, chosen to exclude ones that have already been described, ones with multiple nonlinearities, and ones that are a superset of a simpler equation listed in the table. These cases have not been carefully verified, and no values are given for the coefficients *a* and *b* that give chaotic solutions, except that they are positive. The coefficients are arbitrarily put into the leading terms. They are presented here to encourage further study and experimentation. Note that many of the cases are conservative and those cases require a careful choice of initial conditions and tend to have very small Lyapunov exponents.

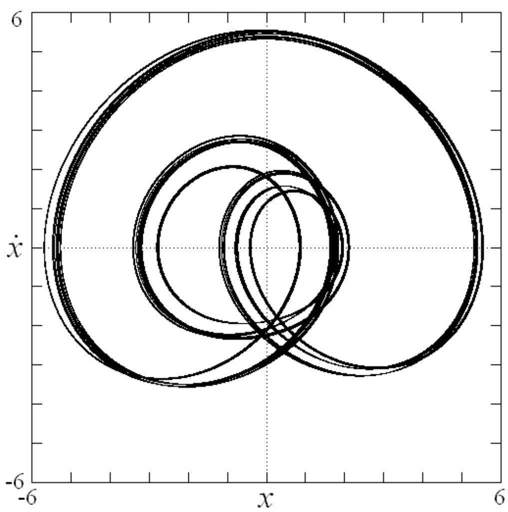

Figure 16. Attractor for the least nonlinear chaotic system given by Eq. (12) with a = 0.025 and $G(x)$ = 0.468|x| - 1

6. Electrical Circuit Implementations

The piecewise linear jerk functions described above are ideally suited for electronic implementation [49] because they can be accurately represented with resistors, capacitors, diodes and operational amplifiers. The general procedure is to start with the quantity $-\dddot{x}$ at the circuit input, and then successively generate $\ddot{x}$, $-\dot{x}$, and x with inverting integrators. These signals, perhaps with an additional constant voltage, are then appropriately combined to form the function $J(\ddot{x}, \dot{x}, x)$, which is then fed back to the input of the circuit. Such a circuit can be considered a nonlinear oscillator with positive feedback.

Table 3. Some additional simple systems with chaotic solutions for appropriate values of *a* and *b* (not given).

$\dddot{x} = a\ddot{x} - b\,\mathrm{sgn}(\ddot{x}) - \dot{x} - x$
$\dddot{x} = -a\dot{x} - bx + \cosh(x) - 1$
$\dddot{x} = -a\dot{x} \pm (\lvert x \rvert - 1)$
$\dddot{x} = -a\dot{x} \pm (x - x^3)$
$\dddot{x} = -a\dot{x} \pm b(x - \max(x,0) + 1)$
$\dddot{x} = -a\dot{x} \pm b(x - \min(x,0) - 1)$
$\dddot{x} = -a\dot{x} \pm (x - \sinh(x))$
$\dddot{x} = -a\dot{x} \pm x + x^2$
$\dddot{x} - -a\dot{x} \pm bx - \cosh(x) + 1$
$\dddot{x} = -a\ddot{x} + b\dot{x} - \dot{x}^3 - x$
$\dddot{x} = -a\ddot{x} + b\dot{x} - \sinh(\dot{x}) - x$
$\dddot{x} = -a\ddot{x} - \exp(\dot{x}) - x$
$\dddot{x} = -a\ddot{x} - b\dot{x} + \cos(\dot{x}) - x$
$\dddot{x} = -a\ddot{x} - b\dot{x} + x^2 - 1$
$\dddot{x} = -a\ddot{x} - b\dot{x} + x - \cosh(x)$
$\dddot{x} = -a\ddot{x} - b\dot{x} + x \pm \exp(x)$
$\dddot{x} = -a\ddot{x} - b\dot{x} - \min(x,0) - 1$
$\dddot{x} = -a\ddot{x} - b\dot{x} - x \pm \cosh(x)$
$\dddot{x} = -a\ddot{x} - b\dot{x} \pm (\cosh(x) - 1)$
$\dddot{x} = -a\ddot{x} - b\dot{x} \pm \cosh(\dot{x}) - x$
$\dddot{x} = -a\ddot{x} - b\dot{x} \pm x - x^2$
$\dddot{x} = -a\ddot{x} \pm b(\cosh(\dot{x}) - 1) - x$

As an example, Fig. 17 shows a circuit that solves Eq. (30) with b = 1. In this and the following circuit, only the inverting input to the operational amplifiers are shown; the noninverting inputs are grounded. If the fixed resistors are 1 Ù, the capacitors are 1 F, and the battery is 1 V, the circuit should work in real time and should produce chaotic oscillations when the variable resistor is adjusted to a value of 1/a 1.67 Ù. However, the frequency at the first Hopf bifurcation at a = 1 is only 1/2ð Hz. A more practical implementation uses resistors of 1 kÙ and capacitors of 0.1 ìF, giving a fundamental frequency of f = 10^4/2ð 1592 Hz at the first bifurcation. This frequency is well into the audio range so that the period doublings, periodic windows, and chaos, as shown in Fig. 13, are easily heard in the signal $x(t)$. The period doublings are even more pronounced when the signal x is integrated before amplification to enhance audibly the low frequencies.

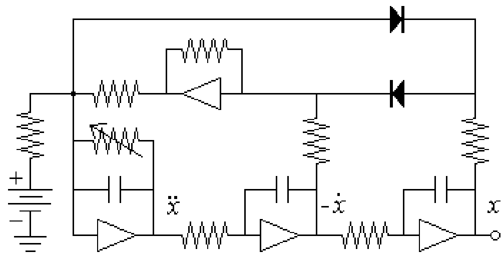

Figure 17. A chaotic circuit using inverting operational amplifiers and ideal diodes that solves Eq. (30) with b = 1.

The operation of the circuit in Fig. 17 should be apparent to anyone with operational amplifier design experience. However, it is not the simplest circuit that solves Eq. (12) with $G(x)$ as shown in Fig. 18 (a). One of the active integrators can be replaced with a passive integrator, and one of the diodes can be eliminated, resulting in a circuit with fifteen components rather than eighteen [50]. Elwakil and Soliman [51] have also devised a chaotic operational amplifier circuit with fifteen components using resistors, capacitors, and diodes, but the equations required to model it are much more complicated.

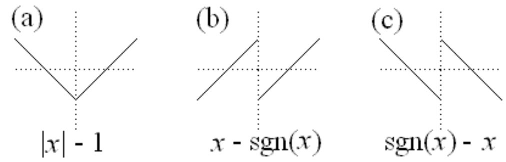

Figure 18. Some functions $G(x)$ in Eq. (12) that lead to chaos and are easily implemented electronically.

The functions shown in Table 2 suggest other nonlinear circuits. For example, the function sgn(x) is easily implemented with an operational amplifier that has no feedback and thus acts as a comparator, abruptly switching output from a large positive to a large negative value as the input voltage crosses zero. Special operational amplifier comparators are available that have orders of magnitude better frequency response and slew rates than those designed for linear operations. Figure 19 shows a circuit that solves the equation

$$\dddot{x} + a\ddot{x} + \dot{x} - x + \mathrm{sgn}(x) = 0 \qquad (34)$$

with a = 0.5, whose nonlinearity is of the form shown in Fig. 18 (b). In this circuit, capacitors are in microfarads and all resistors are 1 kÙ.

This circuit has eleven components and might be the simplest chaotic circuit using only inverting operational amplifiers, resistors and capacitors. Its attractor is a single folded band similar to the Rössler attractor and other examples previously discussed. Equation (34) with the last two signs reversed as in Fig. 18 (c) also has chaotic solutions, with a double-scroll attractor similar to the Lorenz attractor, as shown in Fig. 20, but its circuit implementation requires an additional inverting amplifier. There is not a direct correspondence between simple equations and simple circuits, although the identification of one may suggest the other.

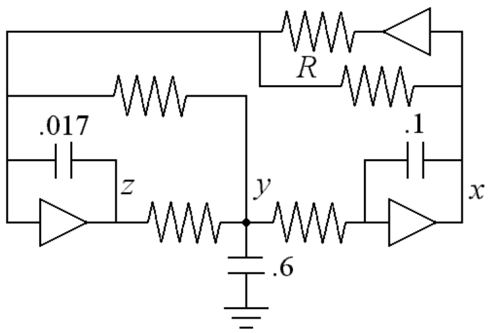

Figure 19. A chaotic circuit using inverting saturating operational amplifiers that solves Eq. (34) with a = 0.5.

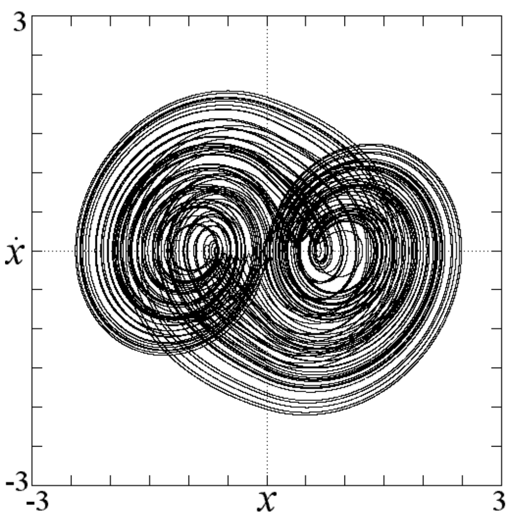

Figure 20. Attractor for the chaotic system given by Eq. (12) with a = 0.5 and $G(x) = \mathrm{sgn}(x) - x$.

For some purposes, such as where size and expense are crucial, it may be important to minimize the required total capacitance and to keep the resistances small so that the whole circuit can be fabricated on a single chip. There is no fundamental reason why this cannot be done, although the challenge would be in simultaneously keeping the frequency low. Note that the initial charge on the three capacitors corresponds to the three initial conditions in the equations. Thus the circuits may not start properly if the initial values lie outside the basin of attraction for the desired solution.

These circuits are similar in spirit to Chua's circuit [52, 53] that uses two capacitors, an inductor, and diodes with operational amplifiers or transistors to provide a piecewise linear approximation to a cubic nonlinearity. A version of Chua's circuit using saturating operational amplifiers is shown in Fig. 21. The equations modelling this circuit can be reduced to

$$\begin{aligned} \dot{x} &= a(y - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -by \end{aligned} \tag{35}$$

which is simple in the sense of having six terms and one nonlinearity, $h(x)$, whose form is similar to $G(x)$ in Eq. (32). However, its jerk representation is much more complicated:

$$\begin{aligned} &\dddot{x} + \ddot{x} + (b - a)\dot{x} = \\ &-a[\ddot{h}(x) + \dot{h}(x) + bh(x)] \end{aligned} \tag{36}$$

Because of the discontinuities in $\dot{h}$ and $\ddot{h}$, the dynamics are not continuous in the space of $(x, \dot{x}, \ddot{x})$. Since the contraction is not constant along the trajectory, it is more difficult to verify the Lyapunov exponents. Chua's circuit is more difficult to construct, scale to arbitrary frequencies, and analyse because of the inductor with its frequency-dependent losses, although a variant of Chua's circuit with only capacitors is possible [54]. Three reactive components (capacitors or inductors) are required for chaos in systems with continuous flows so that the Kirchhoff representation of the circuit contains three first-order ODEs.

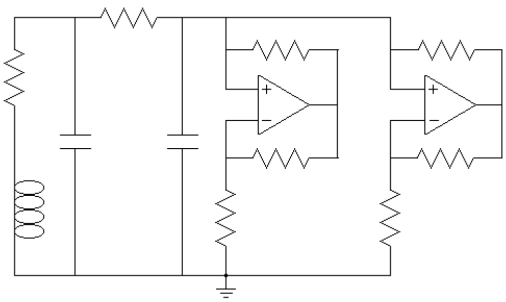

Figure 21. A version of Chua's circuit using saturating operational amplifiers.

Systems involving delta functions (derivatives of the step function) and hysteretic (double-valued) functions can also lead to chaos. Since the flow is discontinuous for such cases, chaos is possible with fewer than three variables. A 2-D chaotic circuit with thirteen components based on this idea has been developed by Tamaševièius, *et al*. [55].

A further 2-D chaotic example is provided by Dixon, *et al*. [56], in which the flow

$$\dot{x} = \frac{xz}{x^2+z^2} - ax$$
$$\dot{z} = \frac{z^2}{x^2+z^2} - bz - (1-b) \quad (37)$$

is singular at the origin and all orbits are forced to approach the singularity. It is probably not the simplest such case. Such examples will not be further discussed because they are usually not good models for natural phenomena.

7. Conclusions

Many autonomous chaotic systems have been discovered and studied that are algebraically simpler than the Lorenz and Rössler systems that are usually cited as prototypical dissipative chaotic flows. The representation of these systems in terms of a single, autonomous, third-order, scalar ODE (a jerk equation) has simplified their identification and classification. Candidate equations have been found for the simplest such systems with quadratic and piecewise linear nonlinearities, Eq. (11) and Eq. (30), respectively. Those systems with piecewise linear functions are especially suited for electronic implementation, and several such circuits have been described. The simplest such circuit may not yet have been identified.

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