Residual energy in MHD turbulence and in the solar wind

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Abstract. Recent observations indicate that kinetic and magnetic energies are not in equipartition in the solar wind turbulence. Rather, magnetic fluctuations are more energetic and have somewhat steeper energy spectrum compared to the velocity fluctuations. This leads to the presence of the so-called residual energy $E_r = E_v - E_b$ in the inertial interval of turbulence. This puzzling effect is addressed in the present paper in the framework of weak turbulence theory. Using a simple model of weakly colliding Alfvén waves, we demonstrate that the kinetic-magnetic equipartition indeed gets broken as a result of nonlinear interaction of Alfvén waves. We establish that magnetic energy is indeed generated more efficiently as a result of these interactions, which proposes an explanation for the solar wind observations.

Keywords: Alfvén waves, Turbulence, Solar Wind
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INTRODUCTION

Magnetohydrodynamics (MHD) provides possibly the simplest framework for describing turbulence observed in the solar wind. When compressibility and the dissipation effects may be neglected, MHD equations have exact solutions in the form of Alfvén waves that propagate (in both directions) along the background magnetic field with the same constant Alfvén speed. The characteristic feature of those Alfvén wave packets is that their magnetic and velocity perturbations, $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$, are either exactly aligned or counteraligned with each other: for wave packets propagating against the direction of the background field, one has $\mathbf{v}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t)$, for wave packets propagating in the direction of the field, $\mathbf{v}(\mathbf{x},t) = -\mathbf{b}(\mathbf{x},t)$. Counterpropagating Alfvén wave packets interact when they overlap, or collide, with each other, e.g. [1, 2].

The symmetry between magnetic and velocity perturbations in individual Alfvén waves may suggest that magnetic and kinetic energies should be in equipartition in the developed MHD turbulence. Indeed, phenomenological theories of MHD turbulence often do not distinguish between the two spectra, assuming (explicitly or implicitly) that they are the same. However, recent observational data indicate that kinetic and magnetic fluctuations in the solar wind turbulence are not in exact equipartition at 1AU. There is slight excess of magnetic energy and magnetic fluctuations have slightly steeper Fourier energy spectrum, e.g., [3, 4, 5, 6, 7, 8]. A possible explanation may be that the solar wind turbulence is in a transient state at 1AU [9]. Another possibility is that the solar wind contains some remnant structures, magnetic field discontinuities or current sheets, advected from the sun and not solely generated by the turbulence [10, 11]. Those
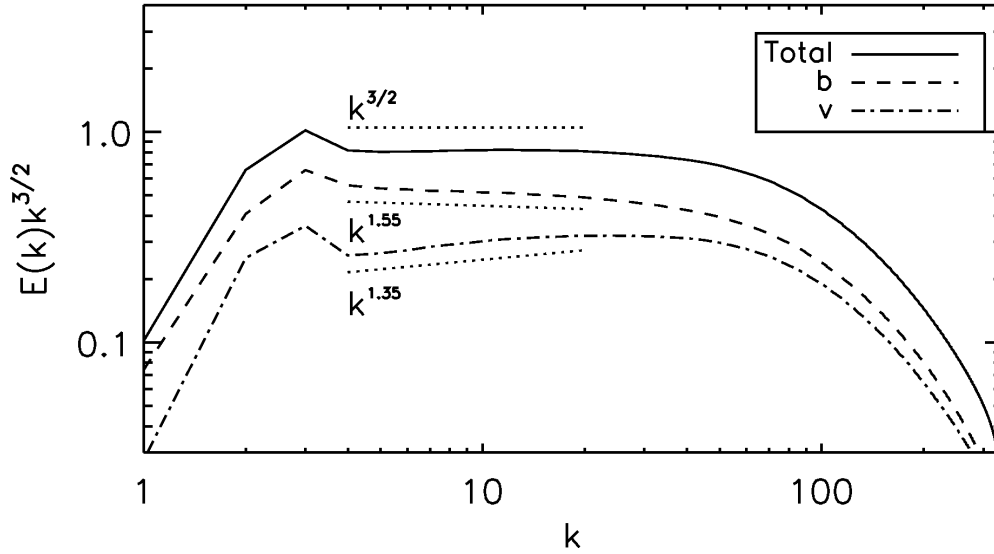
Spectrum of v and b for balanced MHD

FIGURE 1. The field-perpendicular Fourier spectra of kinetic, magnetic, and total energies in numerical simulations of strong incompressible MHD turbulence with a strong guide field. The Elsasser fields z^+ and z^- are driven independently at large scales, so that no residual energy is generated by the driving mechanism. The numerical resolution is $1024^2 \times 256$, the Reynolds number is $Re \approx 5600$. The details of the simulations can be found in [12].

explanations are interesting and may be plausible.

Quite interestingly, recently it has been found that incompressible steady-state MHD turbulence exhibits a very similar mismatch between the kinetic and magnetic energies to that observed in the solar wind [12]. Moreover, it has been discovered that mismatch between magnetic and kinetic energies exists even in the case of *weak* MHD turbulence, that is, turbulence of mostly independent Alfvén waves where equipartition between kinetic and magnetic energies is generally expected [13, 14]. These developments suggest that breakdown of kinetic-magnetic equipartition may, in fact, be a fundamental property of Alfvénic turbulence (rather than an artifact or some nonuniversal plasma effects), which can be understood in the framework of incompressible MHD (see Fig. 1). This effect may be complementary to the previously proposed explanations for the solar wind observations [9, 10, 11].

The difference between kinetic and magnetic energies can be characterized by the so-called residual energy, $E_r = \int (v^2 - b^2) d^3x = \int (\mathbf{z}^+ \cdot \mathbf{z}^-) d^3x$, where $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$. Early studies of MHD turbulence based on some closure assumptions indeed indicated that kinetic and magnetic energies can be different [15, 16, 17]. However, the question remains relatively poorly explored since most of the models of MHD turbulence are based on the quantities like total energy $E = \frac{1}{2} \int (v^2 + b^2) d^3x$ and cross-helicity $H_c = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$, or their combinations, the so-called Elsasser energies $E^\pm = \int (z^\pm)^2 d^3x$, neither of which contains the information about the residual energy. In what follows we present a simple model illustrating how residual energy is generated in a fundamental nonlinear process of MHD turbulence – collision of counter-propagating Alfvén waves.

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A MODEL FOR RESIDUAL ENERGY GENERATION

To formulate the model, we use the standard representation of the incompressible MHD equations in terms of the Elsässer variables,

$$\left(\frac{\partial}{\partial t} \mp \mathbf{v}_A \cdot \nabla\right) \mathbf{z}^\pm + \left(\mathbf{z}^\mp \cdot \nabla\right) \mathbf{z}^\pm = -\nabla P, \qquad (1)$$

where the Elsässer variables are defined as $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$. In these formulas, $\mathbf{v}$ is the fluctuating plasma velocity (the equations are written in a frame with zero mean flow velocity), $\mathbf{b}$ is the fluctuating magnetic field normalized by $\sqrt{4\pi\rho_0}$, $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity corresponding to the uniform magnetic field $\mathbf{B}_0$ (so that the total magnetic field is $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$), $P = p/\rho_0 + b^2/2$ is the total pressure that includes the plasma pressure, p, and the magnetic pressure, ρ_0 is the constant mass density, and we neglect driving and dissipation terms, e.g. [2, 18].

An important fact can be derived from Eq. (1). If we consider field configurations where $\mathbf{z}^\mp(\mathbf{x}, t) \equiv 0$, then an arbitrary function $\mathbf{z}^\pm(\mathbf{x}, t) = F^\pm(\mathbf{x} \pm \mathbf{v}_A t)$ is an exact solution of the nonlinear equations (1). This solution represents a non-dispersive Alfvén wave propagating along the direction of the guide field with the velocity $\mp\mathbf{v}_A$. Nonlinear interactions are thus the result of interactions between counter-propagating Alfvén wave packets. If during one interaction (one crossing time) the deformation of the wave packets is significant, then the turbulence is called strong; if the deformation is weak, the turbulence is called weak. Weak turbulence can be studied using perturbation theory, e.g., [19, 20]. In what follows we study how weak interactions of counter-propagating Alfvén wave packets can generate residual energy, even if the residual energy is absent originally.

Let us assume that the strong guide magnetic field $\mathbf{B}_0$ is in z-direction. We consider our system in a box that is periodic in x and y directions, with the dimensions $2\pi \times 2\pi$, and sufficiently long in the z-direction. When the interaction is neglected, we assume that the waves are shear-Alfvén waves with polarizations normal to the background field, and that they have the following simple field-perpendicular structures:

$$\mathbf{z}_0^+(x, y, z, t) = f^+(z + v_A t) \cos(y)\hat{\mathbf{x}}, \qquad (2)$$

$$\mathbf{z}_0^-(x, y, z, t) = f^-(z - v_A t) \cos(x)\hat{\mathbf{y}}, \qquad (3)$$

The functions (2, 3), supplemented by $P_0 \equiv \text{const}$ that follows from the incompressibility condition, are the exact solutions of Eq. (1) without the nonlinear terms. This form ensures that without the nonlinear interaction, the residual energy is identically zero: $\mathbf{z}_0^+ \cdot \mathbf{z}_0^- = 0$. For simplicity, we also assume that the waves propagating in the opposite directions are not correlated with each other, that is, $\langle f^+ f^- \rangle = 0$. This happens, for example, if the functions f^+ and f^- are random functions of their arguments with certain correlation lengths $l_\parallel$, which can be interpreted as the typical size of the interacting wave packets in the direction of the guide magnetic field $\mathbf{B}_0$. The angular brackets then mean averaging over the z-direction (practically, over scales much larger than $l_\parallel$).

We will consider the case of weak turbulence, that is, $z_0^\pm l_\parallel \ll v_A l_\perp$, where $l_\perp = 2\pi$ is the field-perpendicular size of the interacting wave packets. This condition follows

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from the fact that the linear terms in Eq. (1), estimated as $v_A z_0^\pm / l_\parallel$, must be larger than the nonlinear ones, $z_0^\pm z_0^\mp / l_\perp$. In addition we assume that $l_\parallel \gg l_\perp$, which allows us to retain only the field-perpendicular gradient of the pressure term, and ensures that only shear-Alfvén waves are generated by nonlinear interactions, see, e.g., [20, 21, 22]. Let us now assume that interaction is switched on at time $t = 0$. The solution can then be found perturbatively in small parameter $\varepsilon = z^\pm / v_A$, that is, we can write:

$$\mathbf{z}^\pm = \mathbf{z}_0^\pm + \mathbf{z}_1^\pm + \mathbf{z}_2^\pm + \dots \tag{4}$$

We illustrate the main steps of the derivation using the field $\mathbf{z}^+$. It is convenient to introduce the new variables $\xi = z + v_A t$ and $\tau = t$; then the equation for $\mathbf{z}_1^+$ takes the form:

$$\partial_\tau \mathbf{z}_1^+ = -\left(\mathbf{z}_0^- \cdot \nabla\right) \mathbf{z}_0^+ - \nabla_\perp P_1. \tag{5}$$

The pressure P_1 should ensure incompressibility of $\mathbf{z}_1^+$; it can be found by taking the divergence of both sides of (5). One finds this way that

$$P_1 = \frac{1}{2} f^+ f^- \sin(x) \sin(y). \tag{6}$$

We can now integrate Eq. (5) with respect to time, keeping $\xi = const$, and get:

$$\mathbf{z}_1^+ = G^+(z,t) \left[\frac{1}{2} \cos(x) \sin(y)\hat{\mathbf{x}} - \frac{1}{2} \sin(x) \cos(y)\hat{\mathbf{y}} \right], \tag{7}$$

where

$$G^+(z,t) = \int_0^t f^+(\xi) f^-(\xi - 2 v_A \tau) \, d\tau. \tag{8}$$

In a similar fashion, we can derive the first-order correction $\mathbf{z}_1^-$, which is given by

$$\mathbf{z}_1^- = G^-(z,t) \left[-\frac{1}{2} \cos(x) \sin(y)\hat{\mathbf{x}} + \frac{1}{2} \sin(x) \cos(y)\hat{\mathbf{y}} \right], \tag{9}$$

$$G^-(z,t) = \int_0^t f^+(\eta + 2 v_a \tau) f^-(\eta) \, d\tau, \tag{10}$$

where $\eta = z - v_A t$, $\tau = t$, and the integral in (10) is done assuming $\eta = const$. We can proceed this way to the second and higher orders, calculating pressure corrections at each step. The expressions for the second-order corrections $\mathbf{z}_2^+$ and $\mathbf{z}_2^-$ are bulky; they will not be needed for our current discussion and will not be presented here.

We can now study the evolution of residual energy after the interaction is switched on. Substituting the expansion (4) into the expression for the residual energy, we find to the first nonvanishing order

$$E_r = \langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle \mathbf{z}_1^+ \cdot \mathbf{z}_1^- \rangle = -\frac{1}{4} \langle G^+ G^- \rangle, \tag{11}$$

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where the angular brackets mean averaging in the x, y, and z-directions. It can be checked that the following terms vanish: $\langle \mathbf{z}_0^+ \cdot \mathbf{z}_0^- \rangle = \langle \mathbf{z}_1^\pm \cdot \mathbf{z}_0^\mp \rangle = \langle \mathbf{z}_2^\pm \cdot \mathbf{z}_0^\mp \rangle = 0$. The averaging in formula (11) can be easily performed since the functions f^+ and f^- are not correlated with each other. We therefore derive:

$$\langle G^+ G^- \rangle = \int\limits_0^t \langle f^+(\xi) f^+(\eta + 2v_A\tau) \rangle d\tau \int\limits_0^t \langle f^-(\eta) f^-(\xi - 2v_A\tau) \rangle d\tau. \tag{12}$$

We now assume that the distributions of f^+ and f^- are uniform along the z-direction, that is, their correlation functions depend only on the difference of their arguments:

$$\langle f^\pm(z_1) f^\pm(z_2) \rangle = \kappa^\pm(z_1 - z_2). \tag{13}$$

The correlation functions $\kappa^\pm$ are even, and they decline fast when their arguments exceed the correlation length $l_\parallel$. We also consider Fourier transforms of $f^\pm(z)$, $\kappa^\pm(z)$:

$$f^\pm(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \tilde{f}^\pm(k) \exp(ikz) dk, \quad \kappa^\pm(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \tilde{\kappa}^\pm(k) \exp(ikz) dk, \tag{14}$$

so that their Fourier components are related as:

$$\frac{1}{L} \langle |\tilde{f}^\pm(k)|^2 \rangle = \tilde{\kappa}^\pm(k), \tag{15}$$

where L is the system size in the z-direction ($L \gg l_\parallel$). Then the integrals in (12) can be easily done:

$$\int\limits_0^t \langle f^+(\xi) f^+(\eta + 2v_A\tau) \rangle d\tau = \int\limits_0^t \kappa^+(\xi - \eta - 2v_A\tau) d\tau = \int\limits_0^t \kappa^+(2v_A t - 2v_A\tau) d\tau, \tag{16}$$

$$\int\limits_0^t \langle f^-(\xi - 2v_A\tau) f^-(\eta) \rangle d\tau = \int\limits_0^t \kappa^-(\xi - \eta - 2v_A\tau) d\tau = \int\limits_0^t \kappa^-(2v_A t - 2v_A\tau) d\tau. \tag{17}$$

Now, we are interested in times that are much larger that the time necessary for each wave to shift by a correlation length, that is, we assume $t \gg l_\parallel / v_A$. In other words, we are interested in times exeeding a single collision time of two counter-propagating wave packets. Then, the integration limits in (16) and (17) can be formally extended to infinity:

$$\int\limits_0^t \kappa^\pm(2v_A t - 2v_A\tau) d\tau = \frac{1}{2v_A} \int\limits_0^\infty \kappa^\pm(z) dz = \frac{1}{4v_A} \int\limits_{-\infty}^{+\infty} \kappa^\pm(z) dz = \frac{1}{4v_A} \tilde{\kappa}^\pm(0). \tag{18}$$

According to (15), this expression is positive. We therefore obtain that the residual energy is negative, and it is expressed through the wave correlation functions as:

$$E_r = \langle \mathbf{z}_1^+ \cdot \mathbf{z}_1^- \rangle = -\frac{1}{16v_A^2} \tilde{\kappa}^+(0) \tilde{\kappa}^-(0) < 0. \tag{19}$$

This is the main result of our derivation.

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CONCLUSIONS

An important observation is that the residual energy is always negative, which means that the magnetic energy always exceeds the kinetic one. This result, obtained in the framework of incompressible MHD, provides a qualitative explanation for the solar wind observations. More general discussion indicates that this conclusion is quite robust, in that it is independent of the particular form of the initial z^+ and z^- modes [14]. In addition, direct numerical simulations indicate that it holds for strong turbulence as well.

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